U.T.B.M
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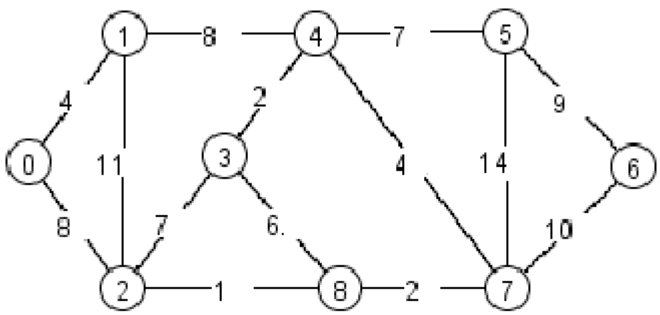
2013-2014
AG44

Final Exam - 2h

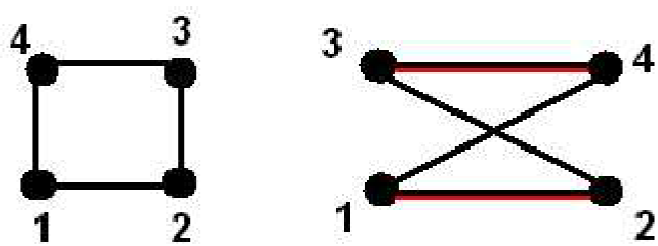
1 Minimum spanning Tree /2

1. Find a minimum spanning tree of the following graph by using an algorithm of your choice (quote the name of the algorithm if it is known). The weight is indicated on each edge.

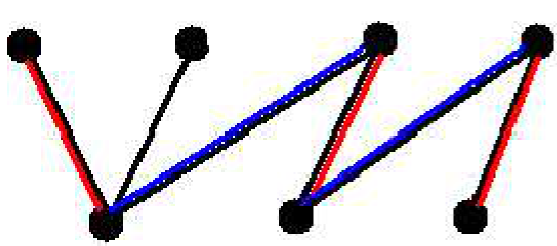


2 Matching /5

Consider a simple bipartite graph $G = (V, E)$. Its vertex set can be partitioned into two disjoint subsets $V = V1 \cup V2$, such that every edge has the form $e = (a, b)$ where $a \in V1$ and $b \in V2$. Note, that no vertices both in $V1$ or both in $V2$ are connected. A subset of edges $M \subset E$ is a matching if no two edges have a common vertex. In the picture below, the matching set of edges is in red :

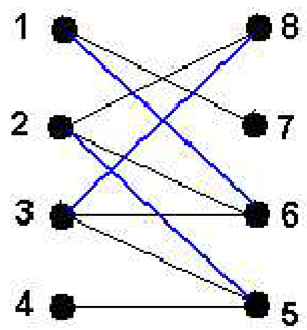


A matching M is maximum, if it has a largest number of possible edges. In this example, blue lines represent a matching and red lines represent a maximum matching.

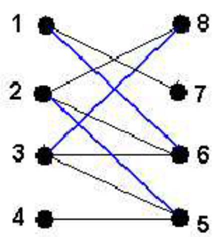


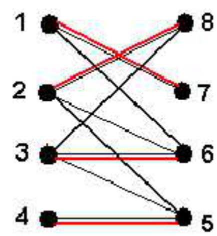
1. What is the maximum number of edges in the maximum matching of a bipartite graph with n vertices ?

A matching M is perfect, if it matches all vertices. We must have $V1 = V2$ in order for a perfect matching to possibly exist. How do we find a perfect matching ? Consider an example. Blue edges represent a matching that is not perfect.



We start at 4, since it has no matching edge, and go to 5. Next we doubt that 2-5 is a part of perfect matching and pair 2 with 8. Again we question 3-8 and pair 3 with 6. One more time, we void 1-6 and instead we pair 1 with 7. Here is our new matching - we change (along the path) blue edges to black and black edges to red to produce the graph on the right.

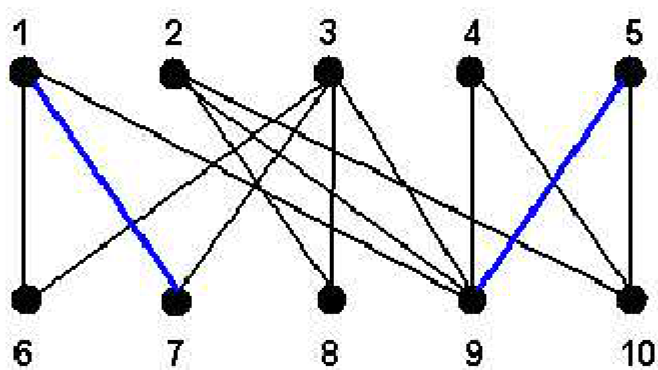




4 - 5 - 2 - 8 - 3 - 6 - 1 - 7

An alternating path is a path whose edges alternate between matched and unmatched edges (like path 4,5,2,8,3,6,1,7)

2. Can we always improve a matching if we find an alternating path (starting from one unmatched vertex and finishing to one matched vertex) ?
3. Suggest an algorithm (not detailed, only the main steps) based on alternating path to obtain a perfect matching. Assume that "Path p = finding-an-alterning-path (vertex v)"' is a function that returns, if exists, an alternating path starting from vertex v.
4. Given a bipartite graph with a matching (graph below), find a perfect matching with your algorithm



5. Give the complexity of your algorithm (in terms of n number of nodes, and m number of edges)
6. Suggest another method to find a perfect matching (try to formulate the problem as a classical graph problem in a particular graph)

3 Mathematical programming /3

We consider the following mathematical programming :

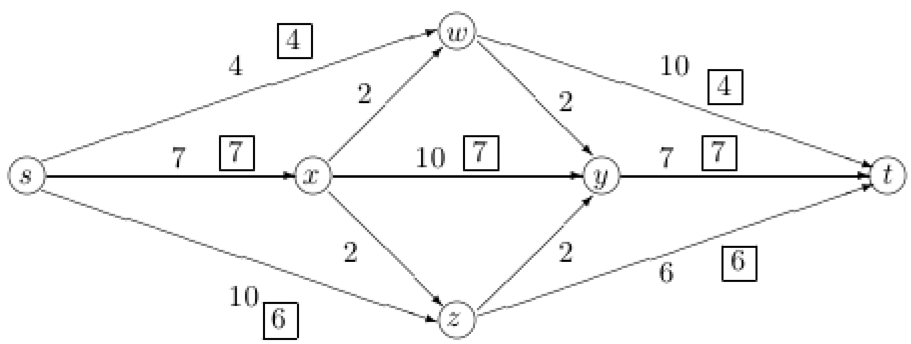
$$
\left\{
\begin{array}{rcccccccccl}
\max f(x_1,x_2,x_3,x_4,x_5) = & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & \\
\text{subject to :} & & & & & & & & & & \\
(1) & x_1 & + & x_2 & & & & & \leq & 1 & \\
(2) & x_2 & + & x_4 & + & x_5 & & & \leq & 1 & \\
(3) & x_1 & + & x_3 & & & & & \leq & 1 & \\
(4) & x_3 & + & x_4 & + & x_5 & & & \leq & 1 & \\
x_i \in \{0,1\} \text{ for } i = 1 \text{ to } 5 & & & & & & & & & &
\end{array}
\right.
$$

Each variable x_i can take value 0 or 1. We seek to maximize the objective function f while satisfying constraints (1) to (4).

1. Model this problem with a undirected graph. Indicate what represents a vertex and an edge in this graph.
2. Show that the problem of maximizing f consists in finding a particular set of vertices in the graph.
3. Give this particular set for the given graph and deduce the maximal value of f.

4 Flow /3

Consider the directed network below. The capacity of each edge appaears as a label next to the edge, and th numbers in boxes give the amount of flow sent on each edge. (Edges without boxed numbers have no flow being sent on them).



1. What is the value of this flow ?
2. Is this a maximum s-t flow in this graph ? If not, find a maximum $s-t$ flow. Which algorithm do you apply ?
3. Find a minimum $s-t$ cut. (Specify which vertices belong to the sets of the cut.)

5 Meeting around a round table /3

One association of 9 players meets every day around a round table. One rule in the association prevents one player from sitting two times nearby the same person.

1. How many days are they able to meet together by satisfying this rule ?
2. Give an organisation of the table for each day.

6 The maximum loaded weight of the truck /4

A road network connects the villages of an island. For each road/edge $e = (u, v)$ of the network, the maximum weight (in tonnes) $p(u, v)$ of the truck that can use that road is known. We try to compute for all pairs of villages u, v, the maximum weight of truck that can go from u to v while respecting the constraints of weight for each possible path from u to v.

1. Formulate the problem by filling the following sentence *Given a graph* $G = (V, E)$ *and a weight* $p : E \to \mathbb{R}$, *we define the capacity of a path* $\sigma = v_1, v_2, .., v_k$ *as* $C(\sigma) = \cdots$. *For two vertices* u *and* v, *we define the capacity* $C(u, v) = \cdots$. *The problem Maximum-Weight-Truck consists in finding* $C(u, v)$ *for each pair of vertices* u, v.
2. Propose an algorithm based on Floyd-Warshall algorithm to solve the problem, which modifications are necessary ? Write the algorithm **Maximum-Weight-Truck** (Give the initial weight matrix that you should consider and the recursive formula to compute the new coefficients of the matrix at each iteration k).