U.T.B.M
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AG44

Final Exam - 2h

1 Minimum spanning tree /6

Part 1

Suppose you have n objects and you define a distance between them. We can represent these distances as edges in a weighted graph. An interesting and useful problem is to group these objects such that objects within a group have a small distance between them, and objects across groups have a large distance between them. This is sometimes called "clustering".

1. One way to compute clusters is to run Kruskal's algorithm, but stop it before the MST has been computed. For example, you could stop it after k edges have been added to T. How many connected components will you have, in that case ?
2. What can you say about the distances of the crossing edges between these components ?

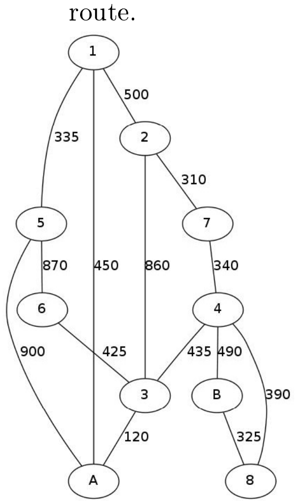
Part 2

Any edge crossing a cut is light edge if its weight is the minimum of all the edge crossing the cut. Light edge is defined with respect to a particular cut. Show that if for every cut of a graph there is a unique light edge crossing the cut, then the graph has a unique minimum spanning tree. Show that the converse is not true by giving a counterexample.
Remark : Do not assume that all weight edges are distinct.

Part 3

The following network represents a portion of a city road network. All the roads are two-way and represented by the edges between two destinations. The numbers on the edges indicate the maximal heights (in centimeters) authorized for the vehicles. A driver wants to make a delivery from point A to point B. In such case, he wants to determine the maximal height x (in meters) of the truck which is authorized.

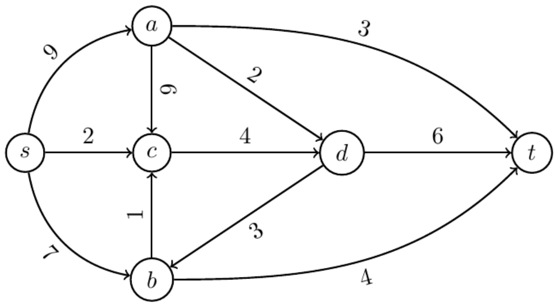
1. Show that this problem involves the resolution of a classical problem of graph. Which one ?
2. Quote and apply an algorithm to solve above problem.
3. Determine the maximal height of the truck, such that the delivery is possible and provide a feasible route.



2 Maximal Flow /4

Consider the network below with source s and sink t. The edge capacities are the numbers given near each edge.

1. Find a maximum flow in this network. (clearly indicate the flow on each edge)
2. Find a minimum s-t cut in the network, i.e. name the two (non- empty) sets of vertices that define a minimum cut.



3 Structure /6

Part 1

In graph theory, the girth of a graph is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles (i.e. it's an acyclic graph), its girth is defined to be infinity. Let G be an n-vertex simple planar graph with girth k. Prove that G has at most $(n-2)\frac{k}{k-2}$ edges. Use this to prove that the Petersen graph is nonplanar.

Part 2

Which of the following is a graphic sequence? If it is graphic, produce a realization of the sequence, else prove why it is not graphic.

1. sequence $(5, 5, 5, 4, 2, 1, 1, 1)$
2. sequence $(5, 5, 4, 4, 2, 2, 1, 1)$

Show by induction on k that for every $k \in N$, the sequence $(1, 1, 2, 2, \ldots, k, k)$ is graphic.

Part 3

Show that if n people attend a party and some shake hands with others (but not with themselves), then at the end, there are least two people who have shaken hands with the same number of people.

4 Modelisation /4

Do only 2 parts among the three ones

Part 1

Suppose you live with $n-1$ other people in an off-campus cooperative apartment. Over the next n nights, each of you is supossed to cook dinner for the entire group exactly once, so that somenone different cooks on each night. Due to scheduling constraints (cobcerts, sports, etc) each person is unable to cook on certain nights, so deciding on who is cooking on each night appears to be a tricky task. Suppose we label the people in the flat $\{p_1, p_2, ..., p_n\}$ and the nights $\{d_1, d_2, \ldots, d_n\}$. Then for each person p_i, there is a set of nights $S_i \subset \{d_1, d_2, \ldots, d_n\}$ where p_i is unable to cook. A feasible dinner schedule is an assignment of each person in the flat to a different night, so that each person cooks on exactly one night, there is somenone cooking on each night, and if p_i cooks on night d_j then $d_j \notin S_i$. Model this problem as a graph problem and give an algorithm to determine if there is a feasible dinner schedule or not.

Part 2

RentCar wants to find a replacement strategy for its cars for a 4-year planning period.
Each year, a car can be kept or replaced. The replacement cost for each year and period is given in the table below. Each car should be used at least 1 year and at most 3 years. Model this problem as a graph problem and find the replacement strategy at minimal cost.

Equipment obtained at start of year	Replacement cost for # years in operation		
	1	2	3
1	4000	5400	9800
2	4300	6200	8700
3	4800	7100	—
4	4900	—	—

Part 3

In a city there are N houses, each of which is in need of a water supply. It costs W_i dollars to build a well at house i, and it costs $C_{ij}$to build a pipe inbetween houses i and j . A house can receive water if either there is a well built there or there is some path of pipes to a house with a well. Model this problem as a graph problem and give an algorithm to find the minimum amount of money needed to supply every house with water.