MT27-2007A-FS02-01

Final Exam 01/25/2008

EXERCISE 1

Let E be the three-dimensional space with an orthonormal referential (O, i, j, k). Let a and b be two positive constants so that $0 < b < a$.

1. Find the area of a disc with radius b, then deduce the value of the integral

$$\int_{-b}^{b} \sqrt{b^2 - x^2}\, dx.$$

2. We consider the plane surface D described by the points M with coordinates (x, y, z) such that

$$x = 0 \text{ and } (y - a)^2 + z^2 \leq b^2.$$

We define a torus T as the solid region of revolution generated by revolving the surface D in the three dimensional space about the axis Oz (think to an inner tube[1]). Sketch its projections on the yz-plane and on the xy-plane.

Now, you will be asked to calculate the volume of T using two different methods.

a. *Calculation using rectangular coordinates*

We assume that $h \in (-b, b)$. Sketch the intersection S_h of the torus T with the plane $z = h$. Calculate the two radii characterizing the surface S_h and find its area. Finally, prove that the volume of the torus is $2\pi^2 ab^2$.

b. *Calculation using cylindrical coordinates*

We parameterize the solid region T using cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$:

$$0 \leq \theta \leq 2\pi,\ a - b \leq r \leq a + b \text{ and } z_1(r) \leq z \leq z_2(r).$$

Consider the section of T with the half yz-plane $x = 0$ and $y \geq 0$, and use the revolution property of T so that to determine the bounds $z_1(r)$ and $z_2(r)$. Then, calculate the volume of the torus using this parametrization.

EXERCISE 2

We consider E a three-dimensional vector space and B the canonical orthonormal basis (e_1, e_2, e_3).

Let u_1 the vector with coordinates $\dfrac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ in the basis B.

[1] Chambre à air

1. Show that the vector u_1 is normalized and build another orthonormal basis $B' = (u_1, u_2, u_3)$ such that u_2 is a linear combination of (u_1, e_1).

2. Let s be the reflection about the axis generated by u_1.

a. Find the matrix A' of s in the basis B'.

b. Find the relation between the matrix A of the reflection s in B and the matrix A'.

c. Calculate the matrix A.

EXERCISE 3.

Consider the vector space E with the orthonormal referential (O, i, j, k). Let S be the surface with equation

$$x^2 + y^2 - z^2 = 1.$$

1. What is the nature of this surface? Find the intersections of S with a plane parallel to the xy-plane and then with the yz-plane.

2. Now, we want to know if the surface S includes some lines.

a. Prove that such a line D cannot be parallel to the xy-plane.

b. Let $A : \begin{pmatrix} p \\ q \\ 0 \end{pmatrix}$ be a point of the xy-plane, $V : \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$ be a vector which is not parallel to the xy-plane, and D the line parallel to V intersecting the xy-plane at A. Find a parametrization of D with parameter z. Then, prove that the line D is included in S if and only if the matrix $B = \begin{pmatrix} a & p \\ b & q \end{pmatrix}$ is an orthogonal matrix.

c. Remark that the intersection between S and the xy-plane is a circle $\mathcal{C}$. Deduce that for any point A lying in $\mathcal{C}$ there exists two orthogonal lines in S passing through A.

d. Have you met buildings looking like the surface S? Do you have any idea about the interest of calculating the lines D in civil engineering?