MT 27 - Autumn 2013

January, the 16th 2013

EXERCISE 1
1. State the second order Taylor series for a function $f(x, y)$ of two variables.
2. Calculate the approximation of the function $\sin(x + y^2)$ at $(x, y) = (0.1, 0.05)$ to an accuracy in the order of 10^{-1}.
3. Consider two $\mathcal{C}^1$-functions $f : \mathbb{R}^p \to \mathbb{R}^m$ and $\phi : \mathbb{R}^n \to \mathbb{R}^p$, state the chain rule for the differential of $fo\phi$. Apply it to the three cases: (a) $n = 1$, $m = p = 2$, (b) $n = p = 2$, $m = 1$, (c) $n = m = 2$, $p = 1$.
4. For a function $f(x, y)$ becoming the function $\phi(u, v)$ under the change of variables $x = \sin(u+v)$ and $y = \sin(u - v)$, show that ϕ is a composition of two functions and calculate the differential $d\phi$.
5. Determine the second order differential $d^2\phi$.
6. Deduce that

$$\frac{\partial^2 \phi}{\partial u^2} - \frac{\partial^2 \phi}{\partial v^2} = 4(\cos^2 u - \sin^2 v)\frac{\partial^2 f}{\partial x \partial y}.$$

7. Find the local extrema of $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 + xy + y^2 + 2x + 3y$.

EXERCISE 2
1. State the formula of change of variable in a double integral.
2. Apply it properly for the change of variable $x + y = u$ and $y = uv$ in the evaluation of

$$\int\int_D e^{\frac{y}{x+y}}\ dxdy$$

where D is the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.
3. Evaluate the double integral

$$\int\int_R x + y\ dxdy$$

where $R = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4 \text{ and } x \leq 0\}$.

EXERCISE 3 Let E the vector space of the polynomials of 2nd degree. Let define the function over $E \times E$ by

$$\varphi(P, Q) = \int_0^{+\infty} P(t)Q(t)e^{-t}\ dt,$$

and the function $h_n(x) = (x^n e^{-x})^{(n)} e^x$ where the exponent (n) means the n-th derivative.
1. Prove that φ is an inner product over E.
2. Show that for $n \in \{0, 1, 2\}$, $h_n \in E$ and that they constitute a basis of E.
3. Prove that this basis is orthogonal for the inner product φ.
4. Deduce an orthonormal basis of E.
5. Apply the Gram-Schmidt procedure to built another orthonormal basis for the inner product φ from the canonical basis.

EXERCISE 4 (Optional) We search the maximum of the product p of the distances to the edges of a triangle ABC of an interior point M.
1. Denoting by a, b, c the lengths BC, CA, AB and x, y, z, A the area of the triangles MBC, MCA, MAB, ABC, prove that

$$d(M, (BC)) = \frac{2x}{a},\ d(M, (CA)) = \frac{2y}{b} \text{ and } d(M, (AB)) = \frac{2z}{c}.$$

2. Deduce that the product p can be expressed as a function of the two variables (x, y),

$$p(x, y) = \frac{8xy(A - x - y)}{abc}$$

and characterize the region D where the maximization occurs.
3. Determine the maximum of p over D.