Final Exam - January, the 16th 2015

Exercise 1 State and prove the pythagorean theorem in an inner product space.

Exercise 2 State the theorem of convergence of $\mathcal{P}$-series[1] and prove it.

Exercise 3 Study the convergence of the series $\sum \ln(1 + \frac{1}{n^2})$.

Exercise 4 We consider the function f defined on $\mathbb{R}$ by

$$f(x) = \frac{\pi - x}{2} \text{ for } x \in]0\,,\,2\pi[,\ f(0) = 0 \tag{1}$$

and extended by 2π-periodicity.

1. Plot the graph of f in the interval $[-2\pi\,,\,4\pi]$. This function is called the sawtooth wave[2].
2. State the general form of the Fourier series of a T-periodic function f together with the expression of its coefficients a_n and b_n.
3. We denote by a_n and b_n the Fourier coefficients of f.
   (a) Justify that $a_n = 0$ for all $n \neq 0$.
   (b) Admitting that $\int_0^{2\pi} x \sin(nx)\,dx = -\frac{2\pi}{n}$ for all $n \in \mathbb{N}^*$. Calculate the coefficient b_n.
4. Prove the pointwise[3] convergence of the Fourier series of f.

Exercise 5 We approximate the sawtooth wave (1) by a linear combination of the three trigonometric polynomials

$$\phi_0(x) = 1,\ \phi_1(x) = \sqrt{2}\sin x \text{ and } \phi_2(x) = \sqrt{2}\sin(2x)$$

belonging to the Fourier basis of 2π-periodic functions. We recall that this family is orthonormal for the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)\ dx.$$

We define the inner product space $\mathcal{V}$ spanned by these trigonometric polynomials.

1. Calculate the orthogonal projection of the function f defined in (1) on the vector space $\mathcal{V}$. You may use the result of Exercise 4 question 3.
2. Let $d(f, \mathcal{V})$ the distance between the function f and the vector space $\mathcal{V}$.
   (a) Find the expression of $d(f, \mathcal{V})$.
   (b) Calculate this distance.

[1] also called Riemann series
[2] onde en dents de scie.
[3] ie at any $x \in \mathbb{R}$

Exercise 6 Let us denote by $\mathcal{B}_0 = (e_1\,,\, e_2\,,\, e_3)$ the canonical basis of the vector space $\mathbb{R}^3$. We consider the matrix

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix} \in \mathfrak{M}_3(\mathbb{R}).$$

1. Verify that A is an orthogonal matrix.
2. Determine the invariant space $\mathcal{D}$ of A.

   In the following, we denote by $\mathcal{P}$ the orthogonal projection on $\mathcal{D}$.
3. Give, without justification, the kernel and the image of $\mathcal{P}$.
4. Find the matrix B of $\mathcal{P}$ in the basis $\mathcal{B}_0$.

   Let f be the linear mapping in $\mathbb{R}^3$ associated to the matrix A in the basis $\mathcal{B}_0$.
5. Prove that f is a rotation and find a vector n of the rotation axis.
6. Let α be the rotation angle of f.
   (a) Calculate $\cos(\alpha)$.
   (b) Calculate $\sin(\alpha)$.
   (c) Deduce the value of α.