MT3E - Autumn 2023

Final Exam - January, the 17th 2024
2 hours

Exercise 1 - Improper Integrals -

1. State the proposition about 'Continuity of an Integral Depending on a Parameter'.
2. **Integration by Parts for an Improper Integral:** Let u and v be two functions of class $\mathcal{C}^1$ on the interval $[a, +\infty[$.
   (a) Suppose that $\lim_{t\to+\infty} u(t)v(t)$ exists and is finite. Prove that the integrals $\displaystyle\int_a^{+\infty} u(t), v'(t)\ \mathrm{d}t$ and $\displaystyle\int_a^{+\infty} u'(t)\, v(t)\ \mathrm{d}t$ are of the same nature.
   (b) In the case of their convergence prove the formula of integration by part :
$$\int_a^{+\infty} u(t)\, v'(t)\ \mathrm{d}t = \big[uv\big]_a^{+\infty} - \int_a^{+\infty} u'(t)\, v(t)\ \mathrm{d}t.$$

Exercise 2 - Determinants -

1. Prove that if a $n \times n$ matrix A is invertible then $det(A) \neq 0$.
2. Let $a \in \mathbb{R}$ and A be the following matrix
$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & a & 1 \\ a & 1 & 0 \end{pmatrix}.$$
   (a) Calculate the determinant of A and determine for which values of a the matrix is invertible.
   (b) Calculate A^{-1} when A is invertible.

Exercise 3 - Series - Justifying your answer, classify the following series into the four categories:

- GD: those such that u_n does not tend towards 0;
- ZD: those which diverge and such that $\lim u_n = 0$;
- AC: those which converge absolutely;
- SC: those which converge, but not absolutely.

Warning: some cases require two demonstrations, for example, to show that $\sum u_n$ is SC, you have to show that $\sum u_n$ converges and that $\sum |u_n|$ diverges.

1. $\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n} + \frac{1}{n^2}\right)$.
2. $\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$.
3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\sqrt{n+1} - \sqrt{n}\right)^2$.

Exercise 4 - Diagonalization -

1. State and prove the 'Diagonalization Theorem' i.e. the necessary and sufficient condition for a matrix A to be diagonalizable.

2. Let
$$A = \begin{pmatrix} 1 & 0 & 0 \\ -9 & 1 & 9 \\ 9 & 0 & -8 \end{pmatrix}.$$

   (a) Determine the eigenvalues of A.

   (b) Give a basis of eigenvectors of A and diagonalise A.

   (c) We want to determine a matrix B such that $B^3 = A$.

      i. Show that if λ is an eigenvalue of B then λ^3 is an eigenvalue of A.

      ii. Show that each eigenvector of B is also an eigenvector of A.

      iii. Determine the eigenvalues of B and their multiplicity.

      iv. Write the characteristic polynomial P_B of B.

      v. We admit that, according to the Cayley-Hamilton theorem, $P_B(B) = 0_{\mathcal{M}_3(\mathbb{R})}$. Deduce B such that $B^3 = A$.

Exercise 5 - Inner Product Spaces -

1. Give the definition of an orthogonal projection.

2. Let $E = \mathbb{R}_n[X]$ with the inner product: For $P = \sum_i a_i X^i$ and $Q = \sum_i b_i X^i$, $(P \mid Q) = \sum_i a_i b_i$. Let $H = \{P \in E \text{ such that } P(1) = 0\}$. We start with $n = 2$.

   (a) Find a basis of H.

   (b) Find an orthonormal basis of H.

   (c) Calculate the distance $d(X, H)$ between the polynomial X and H.

   (d) Generalize this calculation to any $n \in \mathbb{N}$.

3. Let $E = \mathcal{C}([0,1])$ with inner product $(f \mid g) = \int_0^1 f(t)g(t)\,\mathrm{d}t$, and $F = \{f \in E \text{ such that } f(0) = 0\}$.

   (a) Prove that $f \in F^{\perp} \Rightarrow xf \perp f$.

   (b) Show that $F^{\perp} = \{0\}$.

Exercise 6 - Differential Equation -

1. Consider the differential equation $y'(t) = \cos^2(ty)$. Is there existence and uniqueness of solution for each initial condition $(t_0, y_0) \in \mathbb{R}^2$?

2. Solve the differential equation $y'(x) = \dfrac{\sqrt{x-4}}{y^2}$ where $y(4) = -1$.