Final Exam - January, the 13th 2026

Duration: 2 hours

Theory Questions

In your answers to the following questions, you must provide complete and rigorous explanations of all concepts that you use.

1. Prove the following necessary condition for the convergence of a series. Let $\sum u_n$ be a convergent series of real numbers. Then we must have limit of $\lim_{n\to\infty} u_n = 0$.

2. Give the definition of an eigenvalue of an endomorphism.

3. Prove the following characterization of eigenvalues. Let E be a vector space over a field $\mathbb{K}$, let $f \in \mathcal{L}(E)$, and let $\lambda \in \mathbb{K}$: λ is an eigenvalue of $f \iff f - \lambda id_E$ is not a one-to-one mapping.

True/False

For each of the following statements, do not copy the question. Simply answer *true* or *false*, and justify your answer carefully but briefly. A justification of one or two lines is sufficient. Marks will be awarded solely on the quality and relevance of the justifications.

1. If $\sum |u_n|$ converges, then for any choice of signs $\varepsilon_n \in \{-1, 1\}$, the series $\sum \varepsilon_n u_n$ converges.

2. If $u_n = a_n - a_{n+1}$ and if (a_n) does not have a finite limit, then the series $\sum u_n$ cannot converge.

3. Let A be a square matrix, if $\det(A) = 0$, then there exists a vector $X \neq 0$ such that $AX = 0$.

4. If two matrices A and B satisfy $\det(A) = \det(B)$, then they are necessarily similar.

5. If $C_3 = C_1 + C_2$ (columns), then $\det(C_1, C_2, C_3) = 0$.

6. If A is invertible, then $\det(ABA^{-1}) = \det(B)$.

7. For an $n \times n$ matrix, one has $\det(\lambda A) = \lambda \det(A)$.

Exercise 1

Let $E = \mathbb{R}_2[X]$ be the vector space of real polynomials of degree at most 2, equipped with the inner product

$$\langle p, q\rangle = \int_0^1 p(x)\, q(x)\, dx.$$

Let $U = \operatorname{span}(1, X)$.

1. Construct an orthonormal basis (e_1, e_2) of the subspace U. It is recommended to check your result by computing the norms and the inner product of the basis vectors, since this result is used in the following questions.

2. Recall the definition of the *orthogonal projection* onto a subspace U of an inner product space E.

3. Let P_U denote the orthogonal projection from E onto U. Using only the definition of an orthogonal projection, prove that
$$P_U^2 = P_U.$$

4. Compute the image of the polynomial X^2, that is, determine $P_U(X^2)$.

5. Show that the matrix of P_U with respect to the standard basis
$$\mathcal{B} = (1, X, X^2)$$
of E is
$$\mathrm{Mat}_{\mathcal{B}}(P_U) = \begin{pmatrix} 1 & 0 & -\frac{1}{6} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

6. Using the property in Question 3 of an orthogonal projection, deduce an annihilating polynomial for the linear operator P_U.

7. Determine the eigenvalues of P_U and state the dimension of each corresponding eigenspace.

8. Solve explicitely the homogeneous differential equation
$$Y'(t) = P_U\big(Y(t)\big),$$
where $Y(t)$ takes values in the vector space E.

9. Find the solution satisfying the initial condition
$$Y(0) = X^2.$$
Describe the behaviour of the solution using the orthogonal decomposition
$$E = U \oplus U^{\perp}.$$

Exercise 2

This exercise shows how to use convergence of improper integrals to extend existence and uniqueness of the solution to a differential equation given by the Cauchy-Lipschitz theorem given in class.

We consider the differential equation defined on $(0, +\infty)$:

$$(E) \qquad y'(t) + y(t) = f(t), \qquad f(t) = \frac{1}{t^\alpha (1+t)^\beta}, \quad \alpha, \beta \in \mathbb{R}.$$

1. Discuss, depending on the parameter α, the existence and uniqueness of a solution of (E) in a neighbourhood of a point $t_0 > 0$, and then on an interval of the form $]0, \varepsilon]$ with $\varepsilon > 0$. Clearly state which assumption of the theorem of existence and uniqueness of the solution of a first order linear differential equation is involved.

2. Let $t_0 > 0$. Let's recall the Duhamel formula, namely that any solution y of (E) satisfies, for $t > t_0$,
$$y(t) = e^{-(t-t_0)}y(t_0) + \int_{t_0}^{t} e^{-(t-s)} f(s)\, ds.$$
Show that the existence and uniqueness of a solution defined up to $t = 0$ is related to the convergence of an improper integral. Identify this integral explicitly, and study its convergence near $t = 0$ depending on the parameter α. Deduce a condition ensuring the existence of a solution defined on $[0, t_0]$, with $t_0 > 0$.

3. Now, assume that the solution is defined on $[0, +\infty[$. Show that the solution can be written in the form
$$y(t) = e^{-t}y(0) + \int_{0}^{t} e^{-(t-s)} f(s)\, ds.$$