FINAL EXAM

An electronic calculator, printed and personnal class notes (lecture and recitation) are allowed for the exam. Any kind of dictionary is permitted.

Exercise 1

The two parts are independent.

Part A: Choosing the good fixed-point method

One considers the following nonlinear equation

$$x^3 + 4x^2 - 10 = 0 \tag{1}$$

1. *Prove that Eq (1) has a unique solution on* $[1, 2]$.

2. *Show that solving Eq (1) is equivalent to solve any of the three following fixed-point problems* $x = g(x)$ *with:*

$$\begin{aligned} g_1(x) &= x - x^3 - 4x^2 + 10 \quad g_2(x) = \frac{1}{2}\sqrt{10 - x^3} \\ g_3(x) &= x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \end{aligned} \tag{2}$$

3. *Does the fixed-point iteration method converge for* g_1, g_2 *and* g_3 *(hint: for this question you can provide an answer based on your calculator)?*

4. *If it converges what is the rate of convergence (linear, quadratic) ?*

Part B: Newton's method for mutliple roots

1. *Let* g *be a function of* $\mathcal{C}^{m+1}([a, b])$ *such that* g *has a unique fixed point* $\alpha \in [a, b]$. *Prove that, if* $g'(\alpha) = \cdots = g^{(m)}(\alpha) = 0$, *then the rate of convergence of the fixed point method is of order (at least)* $m + 1$.

2. *Let us consider a nonlinear equation.*

$$f(x) = 0 \tag{3}$$

where f *is a function in* $\mathcal{C}^1([a, b])$. *We assume Eq. (3) has a unique solution* $\alpha \in [a, b]$. *Find the function* g *such that Newton's method can be written as a fixed-point iteration method* $x_{n+1} = g(x_n)$. *Show that* $g'(x) = \dfrac{f(x)f''(x)}{f'(x)^2}$.

3. *We assume that* α *is a double root of* f, *i.e.* $f(\alpha) = f'(\alpha) = 0$. *It means that* $f(x) = (x - \alpha)^2 h(x)$ *with* $h(\alpha) \neq 0$. *Deduce from the previous that in the case of a double root, the convergence of Newton's method is linear (instead of quadratic in the simple root case).*

4. *Prove that, if* α *is a double root of* f, *then the modified Newton's method* $x_{n+1} = x_n - 2\dfrac{f(x_n)}{f'(x_n)}$ *converges quadratically in the neighborood of* α.

Exercise 2

The goal of this execise is to calculate by a numerical method

$$\int_0^1 \frac{f(x)}{\sqrt{x}} dx \tag{4}$$

All numerical results will be given with a four decimal digit precision.

Part A: Building your own gaussian method

On the space of real polynomial functions $\mathcal{P}$, one considers the following braket

$$\begin{array}{ccc} \mathcal{P}\times\mathcal{P} & \longrightarrow & \mathbb{R} \\ (p,q) & \longmapsto & \langle p,q\rangle = \displaystyle\int_0^1 \frac{p(x)q(x)}{\sqrt{x}}dx \end{array} \tag{5}$$

1. *Show that $\langle,\rangle$ is an inner product.*

2. *What is the norm of the constant polynomial $x \mapsto 1$ with respect to $\langle,\rangle$?*

3. *Show that for any $m, n \in \mathbb{N}$ we have*

$$\langle x^n, x^m\rangle = \frac{2}{2(m+n)+1} \tag{6}$$

*(*Equation 6 can be used for all calculations involving $\langle,\rangle$ in the the following questions*)*

4. *Let $p(x) = x^2 + bx + c$ a degree 2 polynomial. Find the values of the constants b and c to insure that*

$$\begin{cases} \langle p,1\rangle & = & 0 \\ \langle p,x\rangle & = & 0 \end{cases} \tag{7}$$

5. *Deduce that $(x^2 - \frac{6}{7}x + \frac{3}{35}) \perp \mathcal{P}_1$.*

6. *We would like to use our previous calculation to evaluate (4) with a $n+1 = 2$ points method.*

- **a.** *Explain why the good node set to choose for this problem is $\{x_0, x_1\}$ with $x_0 = \frac{3}{7} - \frac{2}{35}\sqrt{30}$ and $x_1 = \frac{3}{7} + \frac{2}{35}\sqrt{30}$.*
- **b.** *Explain which calculation should be perform to obtain the associated weights W_0 and W_1 (we do not ask to do the calculation explicitly).*
- **c.** *We give $W_0 = 1+\frac{\sqrt{30}}{18}$ and $W_1 = 1-\frac{\sqrt{30}}{18}$. Provide an approximation of $\int_0^1 \frac{\sin(x)}{\sqrt{x}}dx$.*

Part B: Using the Gauss-Legendre method

1. *Find a linear change of variables $u = \alpha x + \beta$ and a new function g such that*

$$\int_0^1 \frac{f(x)}{\sqrt{x}}dx = \int_{-1}^1 g(u)du \tag{8}$$

2. *Use the Gauss-Legendre formula with $n + 1 = 2$ points to provide an approximation of $\int_0^1 \frac{\sin(x)}{\sqrt{x}}dx$.*

3. Matlab *gives the following numerical approximation*

$$\int_0^1 \frac{\sin(x)}{\sqrt{x}}dx \approx 0.6205 \tag{9}$$

Which method, Gauss-Legendre or your own, is better to approximate $\int_0^1 \frac{\sin(x)}{\sqrt{x}}dx$? Explain why one of the two options was not accurate ?

Corrections

Exercise 1

PartA:

1. *Let $f(x) = x^3 - 4x^2 - 10$. The function f is continuous and $f(1) = -5$ and $f(2) = 14$ ($f(1)f(2) \leqslant 0$) so by the mean value theorem one knows there exists $\alpha \in [1, 2]$ such that $f(\alpha) = 0$. Moreover $f'(x) = 3x^2 + 8x > 0$ on $[1, 2]$. Thus f is strictly increasing and therefore there exists a unique $\alpha \in [1, 2]$ such that $f(\alpha) = 0$.*

2. *Let's check that Eq (3) is equivalent to $g_i(x) = x$ for $i = 1, 2$ and 3:*
   - $g_1(x) = x \Leftrightarrow x - x^3 - 4x^2 + 10 = x \Leftrightarrow -x^3 - 4x^2 + 10 = 0 \Leftrightarrow x^3 + 4x^2 - 10 = 0$.
   - $g_2(x) = x \Leftrightarrow \frac{1}{2}\sqrt{10 - x^3} = x \underbrace{\Leftrightarrow}_{\text{because } x\in[1,2]} \frac{1}{4}(10 - x^3) = x^2 \Leftrightarrow 10 - x^3 = 4x^2 \Leftrightarrow x^3 + 4x^2 - 10 = 0$.
   - $g_3(x) = x \Leftrightarrow x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} = x \Leftrightarrow \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} = 0 \underbrace{\Leftrightarrow}_{\text{because } 3x^2+8x\neq 0, \forall x\in[1,2]} x^3 + 4x^2 - 10 = 0$.

3. *One knows that the fixed-point method converges on I if $|g'(x)| < 1$. Let us calculate g_i':*
   - $g_1'(x) = 1 - 3x^2 - 8x$ *(this functions is decreasing $g_1'' < 0$ on $[1, 2]$) and $g_1'(1) = -10$, thus $|g_1'(x)| > 1$ for all $x \in [1, 2]$. The method does not converge.*
   - $g_2'(x) = \frac{-3x^2}{4\sqrt{10 - x^3}}$. *If we plot $y = g_2'(x)$ we can check that $|g_2'(x)| < 1$ on $[1, 1.7]$. But plotting $y = x^3 + 4x^2 - 10$ also indicate that $\alpha \in [1, 1.4]$ thus the fixed point method with g_2 will converge to α for any $x_0 \in [1, 1.7]$.*
   - $g_3'(x) = \frac{(x^3 + 4x^2 - 10)(6x + 8)}{(3x^2 + 8x)^2}$. *It is clear that $g_3'(\alpha) = 0$ (because $\alpha^3+4\alpha^2-10 = 0$) which implies that $|g_3'(x)| < 1$ in the neighborood of α. If we plot $y = g_3'(x)$ we can also check that $|g_3'(x)| < 1$ on $[1, 2]$ thus the fixed point method will converge with g_3 for any choice of $x_0 \in [1, 2]$.*

4. *For $g_2'(x)$ one sees that $g_2'(x) \neq 0$ on $[1, 2]$ thus the convergence is linear. One the other hand $g_3'(\alpha) = 0$ so the convergence is at least quadratic (in fact one may recognize that $x_n = g_3(x_{n-1})$ is nothing but the Newton's method).*

Part B:

1. *Under the hypothesis $g'(\alpha) = \cdots = g^{(m)}(\alpha) = 0$ the Taylor-Lagrange expansion of g at α with order $m + 1$ becomes*

$$\exists\zeta_x \in [a, b], g(x) = g(\alpha) + \frac{1}{(m + 1)!}g^{(m+1)}(\zeta)(x - \alpha)^{(m+1)} \quad (10)$$

Thus if we want to calculate the order of convergence of the fixed point method $x_{n+1} = g(x_n)$ we write

$$x_{n+1} - \alpha = g(x_n) - g(\alpha) = \frac{g^{(m+1)}(\zeta_{x_n})}{(m + 1)!}(x_n - \alpha)^{(m+1)} \quad (11)$$

Therefore

$$\frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^{m+1}} = |\frac{g^{(m+1)}(\zeta_{x_n})}{(m+1)!}| \to_{n\to\infty} |\frac{g^{(m+1)}(\zeta)}{(m+1)!}| \tag{12}$$

We conclude that in this case the fixed point method is of order of convergence $m+1$.

2. *The Newton's method* $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ *can be written as a fixed-point method* $x_{n+1} = g(x_n)$ *if we choose* $g(x) = x - \frac{f(x)}{f'(x)}$. *Differentiating this expression one gets* $g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$.

3. *If f has a double roots we have* $f(x) = (x-\alpha)^2 h(x)$, *thus* $f'(x) = 2(x-\alpha)h(x) + (x-\alpha)^2 h'(x)$ *and* $f''(x) = 2h(x) + 4(x-\alpha)h'(x) + (x-\alpha)^2 h''(x)$. *It leads to*

$$g'(x) = \frac{(x-\alpha)^2 h(x)(2h(x) + 4(x-\alpha)h'(x) + (x-\alpha)^2 h''(x))}{(2(x-\alpha)h(x) + (x-\alpha)^2 h'(x))^2} \tag{13}$$

And thus

$$g'(\alpha) = \frac{2h(\alpha)^2}{4h(\alpha)^2} = \frac{1}{2} \neq 0 \tag{14}$$

which implies (according to **1.***) that the fixed point method converges linearly.*

4. *We consider the modified Newton's method* $x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$. *It is equivalent to a fixed-method* $x_{n+1} = g(x_n)$ *for* $g(x) = x - 2\frac{f(x)}{f'(x)}$. *But* $g'(x) = 1 - 2\frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = -1 + 2\frac{f(x)f''(x)}{f'(x)^2}$. *Now* $g'(\alpha) = -1 + 2\frac{2h(\alpha)^2}{4h(\alpha)^2} = -1 + 1 = 0$ *which implies the convergence is at least quadratic.*

Exercise 2

Part A

1.

2. $||1|| = \sqrt{2}$.

3. $\langle x^m, x^n\rangle = \int_0^1 \frac{x^m x^n}{\sqrt{x}}dx = \int_0^1 \frac{x^{m+n}}{\sqrt{x}}dx = \int_0^1 x^{m+n-1/2}dx = [\frac{1}{m+n+1/2}x^{m+n+1/2}]_0^1 = \frac{1}{m+n+1/2} = \frac{2}{2(m+n)+1}$.

4. *The resolution of the system*

$$\begin{cases} \langle x^2+bx+c, 1\rangle &= 0 \\ \langle x^2+bx+c, x\rangle &= 0 \end{cases} \tag{15}$$

leads to $b = -\frac{6}{7}$ *and* $c = \frac{3}{35}$.

5. $p = x^2 - \frac{6}{7}x + \frac{3}{35}$ *is orthogonal to* 1 *and* x *and therefore by linearity orthogonal to any polynomial* $\alpha x + \beta$*. Thus* $p \perp \mathcal{P}_1$.

6. *Because* x_0 *and* x_1 *are the roots of a degree* 2 *polynomial orthogonal to* $\mathcal{P}_1$ *with respect to* $\langle , \rangle$.

7. *Let* $l_0 = \frac{x - x_1}{x_0 - x_1}$ *and* $l_1 = \frac{x - x_0}{x_1 - x_0}$*, then we have* $W_0 = \int_0^1 \frac{l_0(x)}{\sqrt{x}} dx$ *and* $W_1 = \int_0^1 \frac{l_1(x)}{\sqrt{x}} dx$

8. $W_0 \sin(x_0) + W_1 \sin(x_1) \approx 0.6203$.

Part B

1. $\alpha = 2$, $\beta = -1$*. Thus* $x = \frac{u+1}{2}$ *and* $dx = \frac{du}{2}$ *which leads to* $\int_0^1 \frac{f(x)}{\sqrt{x}} dx = \int_{-1}^1 \frac{f(\frac{u+1}{2})}{\sqrt{\frac{u+1}{2}}} \frac{du}{2}$.

It gives $g(u) = \frac{f(\frac{u+1}{2})}{2\sqrt{\frac{u+1}{2}}}$.

2. *With the Gauss Legendre formula we have*

$$\int_{-1}^1 g(u)du = 1g(-\frac{1}{\sqrt{3}}) + 1 \times g(\frac{1}{\sqrt{3}}) = \frac{\sin(\frac{-1/\sqrt{3}+1}{2})}{2\sqrt{\frac{-1/\sqrt{3}+1}{2}}} + \frac{\sin(\frac{1/\sqrt{3}+1}{2})}{2\sqrt{\frac{1/\sqrt{3}+1}{2}}} \approx 0.6276 \quad (16)$$

3. *Numerically our own method is more accurate than Gauss-Legendre. It makes sense because the function* g *is not continuous en* $[-1, 1]$ *and we should not have used the Gauss-Legendre formula.*