FINAL EXAM

An electronic calculator, printed and personnal class notes (lecture and recitation) are allowed for the exam. Any kind of dictionary is permitted.

Exercice 1 Steffensen's method (8 points)

In this problem we consider a nonlinear equation of type

$$f(x) = 0 \tag{1}$$

where f is a function in $\mathcal{C}^3([a,b])$. We assume Eq (1) has a unique solution $\alpha \in [a,b]$ and $f'(\alpha) \neq 0$. The goal of this exercise is to introduce a new numerical method called the Steffensen's method.

1. Technical result:

a. Show that $\lim_{x\to\alpha} \dfrac{f(x)}{f(x+f(x))-f(x)} = \dfrac{1}{f'(\alpha)}$
Hint: you can (should) use the Hospital's rule which says that for two differentiable functions h_1 and h_2 with $h_1(a) = h_2(a) = 0$ then $\lim_{x\to a} \dfrac{h_1(x)}{h_2(x)} = \lim_{x\to a} \dfrac{h_1'(a)}{h_2'(a)}$.

b. Deduce that the function $h : x \mapsto \dfrac{f^2(x)}{f(x+f(x))-f(x)}, \forall x \neq 0$ is continous in α if we assume $h(\alpha) = 0$. We will assume without proving it that h is $\mathcal{C}^2$ on $[a,b]$.

2. Show that α is a fixed point of g defined by $g(x) = x - h(x)$.

3. Prove that $g'(\alpha) = 0$.

4. Conclude that there should be an interval $(\alpha-\varepsilon, \alpha+\varepsilon)$ such that the following method:

$$\begin{cases} x_0 \in (\alpha-\varepsilon, \alpha+\varepsilon) \\ x_{n+1} = x_n - \dfrac{f^2(x_n)}{f(x_n+f(x_n))-f(x_n)} \end{cases} \tag{2}$$

converges to α.

5. Numerical application: one considers in this question $f(x) = x^2 - 2$ and $\alpha = \sqrt{2}$.

a. Write the Steffensen's method in this particular case. We will consider $x_0 = 2$.

b. Compute x_1 and x_2.

c. The following table gives the values calculated for x_3, x_4 and x_5.Compare the values obtained with the numerical approximation of $\sqrt{2} \approx 1.41421356$. What is your opinion about the rate of convergence of the method ?

x_0	x_1	x_2	x_3	x_4	x_5
2	x_1	x_2	1.4191773	1.41424667	1.41421356

6. Rate of convergence:

a. Apply the Taylor-Lagrange formula to g between x and α and show that $g(x) - \alpha = \dfrac{g''(\theta)}{2}(x-\alpha)^2$ where θ be a real number between α and x.

b. Use the previous calculation to calculate the rate of convergence of Steffensen's method.

Exercice 2 Gauss-Lobatto (10 points)

In this exercise we study an alternative to the Gauss-Legendre method called the Gauss-Lobatto method. Like Gauss-Legendre, the Gauss-Lobatto quadrature is a method to provide an approximation of integrals of type

$$I = \int_{-1}^{1} f(x)dx \tag{3}$$

The only difference is that the first and last node points are fixed to -1 and 1. In other words the node set of the Gauss-Lobatto method, for an integration with $n+1$ nodes, is always of the form $\{-1, x_1, x_2, \ldots, x_{n-1}, 1\}$. We start by studying an example and then we discuss the quality of this new method. We will denote $I^{n+1}_{Lobatto}(f)$ the numerical approximation of I by the Gauss-Lobatto method.

Part A: Comparison of Gauss-Legendre ($n+1=3$) and Gauss-Lobatto ($n+1=4$).

One gives the following node set and weight to perform the Gauss-Lobatto quadrature.

x_0	x_1	x_2	x_3	W_0	W_1	W_2	W_3
-1	$-\sqrt{\frac{1}{5}}$	$\sqrt{\frac{1}{5}}$	1	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{1}{6}$

1. Recall how the weights W_0, W_1, W_2 and W_3 (we do not ask to do the calculation but just write the formula which allows to recover those data).
2. Use the Gauss-Lobatto quadrature to evaluate $\int_{-1}^{1}\cos(x)dx$.
3. Compare with the value obtained by using Gauss-Legendre for $n+1=3$ points and with the exact value.
4. Using the symmetry of the node set and weight show that $I^4_{Lobatto}(x^m) = 0$ if $m = 2k+1$.
5. Compute $I^4_{Lobatto}(x^2)$, $I^4_{Lobatto}(x^4)$ and $I^4_{Lobatto}(x^6)$.
6. What is the order the Gauss-Lobatto quadrature with $n+1=4$ points ?

Part B: One denotes by (L_n) the sequence of Legendre polynomials as defined in the class notes. We recall that (L_n) is a familly of orthogonal polynomials for the inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$.Let (Q_n) be a sequence of real polynomials defined by

$$Q_{n+1} = L_{n+1} - \alpha_{n+1}L_n - \beta_{n+1}L_{n-1} \tag{4}$$

where we assume that $\alpha_{n+1}, \beta_{n+1}$ are chosen such that $Q_{n+1}(1) = Q_{n+1}(-1) = 0$.

1. What is the degree of Q_{n+1} ?
2. Prove that $Q_{n+1} \perp \mathcal{P}_{n-2}$ where $\mathcal{P}_k$ denotes the vector space of polynomial of degree at most k.

3. We denote by $\{-1, x_1, \ldots, x_{n-1}, 1\}$ the roots of Q_{n+1}. Those roots are used to built the Gauss-Lobatto quadrature. In this question we will find the order of the Gauss-Lobatto method.

Let $P \in \mathcal{P}_{2n-1}$ and we consider the quadrature formula $I^{n+1}_{Lobatto}(f) = W_0 f(-1) + W_1 f(x_1) + \ldots W_{n-1} f(x_{n-1}) + W_n f(1)$ built from the node set $\{-1, x_1, \ldots, x_{n-1}, 1\}$. We write the euclidiean division of P by Q_n, i.e. there exist polynomials B and R_n such that

$$P(x) = Q_{n+1}(x)B(x) + R_n(x) \tag{5}$$

with $deg(R_n) \leqslant n$. The following questions do not require any calculation.

- **a.** Explain why $I_{Lobatto}(R_n) = \int_{-1}^{1} R_n(x)dx$.
- **b.** What is the degree of B ? Show that $\int_{-1}^{1} Q_n(x)B(x)dx = 0$.
- **c.** Deduce from the previous that $\int_{-1}^{1} P(x)dx = \int_{-1}^{1} R(x)dx$.
- **d.** Prove $I_{Lobatto}(P) = I_{Lobatto}(R_n)$.
- **e.** Conclude that for all $P \in \mathcal{P}_{2n-1}$ we have $\int_{-1}^{1} P(x)dx = I_{Lobatto}(P)$. What is the degree of accuracy of the Lobatto method ?

Corrections

Exercice 1 Steffensen's method (8 points)

1. Technical results:

   a. We just apply the hospital rule. For that we remark that $(f(x+f(x))-f(x))' = f'(x+f(x))(1+f'(x))-f'(x)$. So $\lim_{x\to\alpha}(f(x+f(x))-f(x))' = f'(x+f(x))(1+f'(x))-f'(x) = f'(\alpha)^2$ which gives the expected result.

   b. $\lim_{x\to\alpha} h(x) = f(\alpha) \times \dfrac{1}{f'(\alpha)} = 0$. Thus h is continuous in α if we assume $h(\alpha) = 0$.

2. It is cleas that $g(\alpha) = \alpha - 0$. So α is a fixed point of g.

3. Let us compute $g'(x)$: $g'(x) = 1 - h'(x)$. With $h(x) = f(x) \times \dfrac{f(x)}{f(x+f(x))-f(x)}$ so $h'(x) = f'(x)\dfrac{f(x)}{f(x+f(x))-f(x)} + f(x)(\dfrac{f(x)}{f(x+f(x))-f(x)})'$ Thus $\lim_{x\to\alpha} h'(x) = 1$. Therefore $g'(x) = 0$.

4. $g'(\alpha) = 0$ and g' continous becaus h is $\mathcal{C}^1$. Thus there exists an interval of type $(\alpha-\epsilon, \alpha+\epsilon)$ such that for all $x \in (\alpha-\epsilon, \alpha+\epsilon)$ we have $|g'(x)| \leqslant k < 1$. It insures that the fixed point method is convergent on this interval.

5. Numerical application:

   a.
   $$\begin{cases} x_0 = 2 \\ x_{n+1} = x_n - \dfrac{(x_n^2-2)^2}{(x_n+(x_n^2-2))^2 - 2 - (x_n^2-2)} \end{cases} \tag{6}$$

   b. One gets $x_1 = 1.6666667$ and $x_2 = 1.4774775$.

   c. The rate of convergence seems to be quadratic as the number of exact digits double at each iteration.

6. **a.** Taylor-Lagrange: $g(x) = g(\alpha) + g'(\alpha)(x-\alpha) + \dfrac{g''(\theta)}{2}(x-\alpha)^2$ with θ between x and α. Using $g(\alpha) = \alpha$ and $g'(\alpha) = 0$ one gets $g(x) - \alpha = \dfrac{g''(\theta)}{2}(x-\alpha)$.

   b. Applying the previous equality for x_n and using the relation $x_{n+1} = g(x_n)$ one obtains
   $$x_{n+1} - \alpha = \frac{g''(\theta)}{2}(x_n - \alpha) \Leftrightarrow \frac{x_{n+1}-\alpha}{(x_n-\alpha)} = \frac{g''(\theta)}{2} \longrightarrow_{x\to\alpha} \frac{g''(\alpha)}{2} \tag{7}$$
   Thus if $g''(\alpha) \neq 0$ the method is quadratic.

Exercice 2 Gauss-Lobatto (12 points)

Part A:

1. $W_i = \int_{-1}^{1} l_i(x)dx$ where $l_i(x) = \Pi_{j=0,j\neq i}^{n} \dfrac{x-x_j}{x_i-x_j}$ is the ith Lagrange polynomial for the node set $\{-1, x_1, \ldots, x_{n-1}, 1\}$.

2. $I^4_{Lobatto}(\cos(x)) = \frac{1}{6}\cos(-1) + \frac{5}{6}\cos(-\sqrt{1/5}) + \frac{5}{6}\cos(\sqrt{1/5}) + \frac{1}{6}\cos(1) = 1.68286.$

3. If we compute $I^3_{Legendre}(\cos(x)) = 5/9 * \cos(-\sqrt{3/5}) + 8/9 * \cos(0) + 5/9 * \cos(\sqrt{3/5}) = 1.6830035$. The calculation of I gives $I = 2\sin(1) \approx 1.682941969615793$. It seems like the Gauss-Lobatto quadrature with 4 node points provides a better approximation than the Gauss-Legrendre with 3 points.

4. Because the node set is symmetric with respect to 0 it is clear that $I_{Lobbato}(x^m) = 0$ when $m = 2k + 1$.

5. $I^4_{Lobatto}(x^2) = 0$, $I^4_{Lobatto}(x^4) = 0$ and $I^4_{Lobatto}(x^6) \neq 0$.

6. Thus $I^4_{Lobatto}(x^m) = 0$ for $m \leqslant 5$ and $I^4_{Lobatto} = 5$.

Part B

1. $Q_4 = (x^2 - 1)(x^2 - 1/5)$.

2. Q_{n+1} is of degree $n + 1$ because L_{n+1} is of degree $n + 1$.

3. For all $P \in \mathcal{P}_{n-2}$, one have $\langle Q_{n+1}, P\rangle = \langle L_{n+1} - \alpha_{n+1}L_n - \beta_{n+1}L_{n-1}, P\rangle = \langle L_{n+1}, P\rangle - \alpha_{n+1}\langle L_n, P\rangle - \beta_{n+1}\langle L_{n-1}, P\rangle = 0 - \alpha_{n+1} \times 0 - \beta_{n+1} \times 0 = 0$.

4.

- **a.** Because R is of degree n and a method with $n + 1$ node point is always at least of order n.
- **b.** $\deg(B) \leqslant 2n - 1 - (n + 1) = n - 2$. Thus $\langle Q_{n+1}, B\rangle = 0$.
- **c.** Clear by using the euclidian division
- **d.** $I_{Lobatto}(P) = I_{Lobatto}R$ (just replace in each formula and use the fact that the $xi's$ are roots of Q_{n+1}).
- **e.** Gathering the equality one get $\int_{-1}^{1} P(x)dx = I^{n+1}_{Lobatto}(P)$, i.e. the method is of order $2n - 1$.