MTC7 - Autumn 2019

Final Exam - January, the 17th 2020

Exercise 1 Let $I =]a, b[$ and J two intervals of $\mathbb{R}$ and $f : J \times I \to \mathbb{R}$. Give the conditions on the function f such that the integral depending on a parameter

$$F(x) = \int_a^b f(x,t)\ dt$$

converges and that the function F is continuous over J.

Exercise 2 Prove the statement : *For f a continuous, non-negative and decreasing function on the interval $[1\,,\,+\infty[$, let us pose $u_k = f(k)$ for all integer $k \geq 1$, then*

$$\sum u_k \text{ converges } \iff \int_1^\infty f(x)\ dx \text{ converges.}$$

Exercise 3 In some cases, an invertible $n \times n$ matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$ are such that the solution $x \in \mathbb{R}^n$ of the linear system

$$Ax = b$$

is not stable under small perturbations of b. For instance for $A = \begin{pmatrix} -32 & -66 \\ 66 & 133 \end{pmatrix}$ and $b = \begin{pmatrix} 40 \\ -80 \end{pmatrix}$ the solution is $x = \begin{pmatrix} 0.4 \\ -0.8 \end{pmatrix}$. If we perturb slightly the right-hand side, $b' = \begin{pmatrix} 40.2 \\ -80.1 \end{pmatrix}$ the new solution turns out to be $x' = \begin{pmatrix} 0.6 \\ -0.9 \end{pmatrix}$. In other words, a relative error of the order $0.2/40 = 5\times10^{-3}$ and $0.1/80 = 1.25\times10^{-3}$ in the data b produces a relative error of the orders $(0.6-0.4)/0.4 = 5\times10^{-1}$ and in the first component of the solution, which represents an amplification of the relative error of the order $5 \times 10^{-1}/(5 \times 10^{-3}) = 100$. In this exercise you will discover the reason of this phenomena, and in the same time the spectral norm $||.||_\infty$ of matrices.

a. Posing $b' = b + \delta b$ and $x' = x + \delta x$ prove that $A\delta x = \delta b$.

b. Let $||x|| = \sqrt{x_1^2 + x_2^2}$ be the euclidean norm of any vector $x = (x_1, x_2)^T \in \mathbb{R}^2$ and let the circle $C = \{x \in \mathbb{R}^2 \mid ||x|| = 1\}$. Prove that if $||Ax|| = 0$ for all $x \in C$ then A is the null matrix.

c. Let $\mathcal{M}_2(\mathbb{R})$ be the vector space of the 2×2 matrices and the mapping

$$||.||_\infty : \begin{array}{ccc} \mathcal{M}_2(\mathbb{R}) & \to & \mathbb{R} \\ A & \mapsto & ||A||_\infty = \sup_{x\in C} ||Ax|| \end{array}.$$

Prove that $||.||_\infty$ defines a norm over $\mathcal{M}_2(\mathbb{R})$. To prove the triangle inequality we admit the property of the suppremum over a set E: $\sup_{x\in E}(g(x) + h(x)) \leq \sup_{x\in E} g(x) + \sup_{x\in E} h(x)$ for any functions g and $h : E \to \mathbb{R}$.

d. Prove that the sum of the dimensions of the eigenspaces of a diagonalizable and invertible matrix $A \in \mathcal{M}_2(\mathbb{R})$ is equal to two and that its eigenvalues are non vanishing.

In the following, we consider only diagonalizable and invertible matrices A, we denote λ_1 and λ_2 the eigenvalues corresponding to their two independent eigenvectors U_1 and U_2 and we choose $0 < |\lambda_1| \leq |\lambda_2|$.

e. Prove that $||A||_\infty = |\lambda_2|$. We suggest to use, after its justification, the decomposition $x = aU_1 + bU_2$.

f. Express the eigenvalues of A^{-1} in terms of λ_1 and λ_2 to prove that $||A^{-1}||_\infty = \dfrac{1}{|\lambda_1|}$.

g. Establish that $||By|| \leq ||B||_\infty \, ||y||$ for any $y \in \mathbb{R}^2$ and any diagonalizable and invertible matrix B to prove that

$$\frac{||\delta x||}{||x||} \leq ||A||_\infty ||A^{-1}||_\infty \frac{||\delta b||}{||b||}.$$

An intermediary step is the proof of $\frac{1}{||x||} \leq ||A^{-1}||_\infty \frac{1}{||b||}$.

h. In the introductory example the matrix A is invertible and diagonalizable with $\lambda_1 = 1$ and $\lambda_2 = 100$. Explain the instability result using the ratio $||A||_\infty ||A^{-1}||_\infty = \dfrac{|\lambda_2|}{|\lambda_1|}$ called the *condition number of the matrix* A and propose a condition that garanties a small instability in the resolution of a 2×2 linear system.

Exercise 4 Let a sequence of functions $u_n : x \mapsto \dfrac{(-1)^{n+1}}{(n+x)^2}$ for $n \in \mathbb{N}$ and $x \in]0, +\infty[$.

a. Show that the series $\sum_{n=0}^{\infty} u_n(x)$ simply converges on $]0, +\infty[$.

b. Prove that the series of the derivatives $\sum_{n=0}^{\infty} u_n'(x)$ is normally convergent on $]0, +\infty[$.

c. Find the expression of the derivative $S'(x)$ of the sum $S(x) = \sum_{n=0}^{\infty} u_n(x)$ under the form of a series. A detailed justification is required.

Exercise 5 A door is opened between two rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is assumed to be proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \text{ and } \frac{dw}{dt} = v - w.$$

a. Without computing the solution, show that the total number $v + w$ of people in the two rooms is constant (40 people).

b. Write the system under the form of a system of differential equation $\dfrac{du}{dt} = Au$ with matrix A, solve this system and find v and w at $t = 1$.

Exercise 6 Show but without computation that the minimization problem

$$\min_{a,\, b \in \mathbb{R}} \int_0^1 (e^{2t} - ae^t - b)^2 dt$$

can be expressed as a distance. Be very precise on the definition of this distance.

Exercise 7 Justify in detail the existence of the limit

$$\lim_{n \to \infty} \int_0^\infty \frac{t^n}{1 + t^{n+2}} dt$$

and calculate its value.