MTC7 - Autumn 2020

Final Exam - January, the 15th 2021

Exercise 1

a. State the definition of a uniformly convergent sequence of functions.

b. State the theorem ensuring that the limit of a sequence of continuous functions is continuous.

c. Provide an example of a sequence of continuous functions whose limit is not continuous and show how the previous theorem in **b**. does not apply.

d. State the definition of the determinant $\det_{\mathcal{B}} f$ of an endomorphism f in a basis $\mathcal{B}$.

e. Prove that $\det_{\mathcal{B}} f$ is independent of the basis $\mathcal{B}$.

Exercise 2 Answer each of the following questions by TRUE or FALSE with a **single** sentence of explanation. In case of a FALSE the justification may be through a counter example. What matters is **only** the explanation.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

b. $\displaystyle\sum_{n=1}^{\infty} \tfrac{1}{n}$ diverges.

c. If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

d. If $\lim_{n\to\infty} a_n = 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

e. $\displaystyle\sum_{n=0}^{\infty} (\tfrac{1}{7})^n = \frac{7}{6}$.

f. $\displaystyle\sum_{n=0}^{\infty} (\pi)^n = \frac{1}{1-\pi}$.

Exercise 3 Answer each of the following questions by TRUE or FALSE with a **short** justification. In case of a FALSE the justification may be through a counter example. What matters is **only** the justification.

a. If A is diagonalizable, then A^3 is diagonalizable.

b. If A is a $3\times\ 3$ matrix with eigenvalues $\lambda = 1, 2, 3$, then A is invertible.

c. If A is a $3\times\ 3$ matrix with eigenvalues $\lambda = 1, 2, 3$, then A is diagonalizable.

d. If A is diagonalizable, then it is invertible.

e. A 3×3 matrix A with only one eigenvalue cannot be diagonalizable.

Exercise 4 Construct an orthonormal basis of $\mathbb{R}^2$, using the Gram-Schmidt procedure, for the non-standard inner product $(u, v) = u^T \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} v$ where u^T is the transpose of the vector u.

Exercise 5 Let $b = (2, 3)^T$. Calculate the distance between the vector b and the subspace $V = span\{(1, 1)^T\}$.

Exercise 6 Suppose the curve $y = 1/x$ is rotated around the x-axis generating a sort of funnel or horn shape, called Gabriel's horn or Toricelli's trumpet. Show that the volume of this funnel from $x = 1$ to infinity is finite and compute its volume.

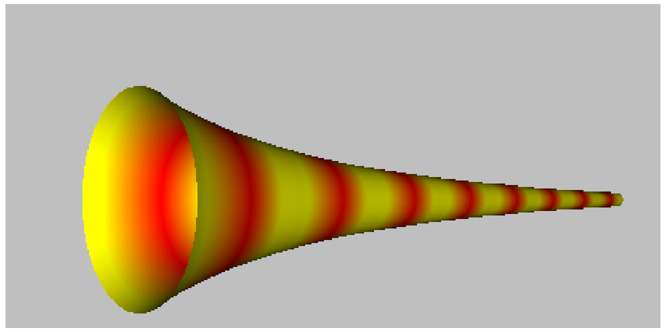

Figure 1: Toricelli's trumpet

Exercise 7 The graph of the gamma function is a smooth curve that connects the points (x, y) given by $y = (x-1)!$ at the positive integer values for x. It satisfies the recurrence relation defining a translated version of the factorial function, $\Gamma(1) = 1$ and $\Gamma(x+1) = x\Gamma(x)$ for any positive real number x. The notation $z \to \Gamma(z)$ is due to the french mathematician Adrien-Marie Legendre (1752-1833). Here, we will recover some properties of the gamma function defined by

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} \, dx$$

for any complex number z with positive real part $\mathrm{Re}(z) > 0$.

a. Prove that for each fixed $z \in \mathbb{R}^{+*}$ the integral $\Gamma(z)$ is convergent.

An improper integral $\int_a^b f(x)\, dx$ of a complex valued function f (i.e. such that $f :]a, b[\to \mathbb{C}$) is said to be absolutely convergent if the improper integral $\int_a^b |f(x)|\, dx$ is convergent, where $|f(x)|$ is the modulus of the complex number $f(x)$.

b. Deduce that the integral $\Gamma(z)$ is absolutely convergent for each fixed complex number z with positive real part.

c. Prove the recurence relation $\Gamma(z+1) = z\Gamma(z)$ for any $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 1$.

d. Deduce that $\Gamma(n+1) = n!$ for integers after proving that $\Gamma(1) = 1$.

e. Prove that the gamma function is continuous over $z \in \mathbb{R}^{+*}$.

f. Prove that the gamma function is derivable for $z \in]1, +\infty[$ and calculate its derivative as an integral.

In probability theory and statistics, the gamma distribution is a two-parameter family of continuous probability distributions. It appears for instance in econometrics and many other applied fields, where for example the gamma distribution is frequently used to model waiting times. For instance, in life testing, the waiting time until death is a random variable that is frequently modeled with a gamma distribution

$$f(x; \alpha, \beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \quad \text{for } x > 0$$

where α and β are the two positive parameters.

g. Use previous results to show that the distribution of probability $x \to f(x; \alpha, \beta)$ is integrable over $\mathbb{R}^+$ for any fixed parameters α and β.

h. Prove that the gamma distribution is actually a continuous probability distributions that is it is non-negative over $x > 0$ and its integral over $\mathbb{R}^+$ is equal to 1.

Exercise 8 The supremum, abbreviated *sup*, of a subset S of $\mathbb{R}$ is the least element in $\mathbb{R}$ that is greater than or equal to all elements of S, if such an element exists. The uniform norm (or sup norm) assigns to any real-valued bounded function f defined on a set A the non-negative number

$$\|f\|_{\infty,A} = \sup \{ |f(x)| \;\mid x \in A \} .$$

Prove that the maping $f \to \|f\|_{\infty,A}$ is actually a norm in the set of continuous functions over a closed interval $A = [a, b]$.