Final Exam - January, the 14th 2021

Exercise 1 All the following statements must be given with much care to the details.

a. State the definition of an improper integral of type 1 and of its convergence.

b. Prove that an absolutely convergent improper integral is convergent.

c. Give the definition of the determinant of an endomorphism and prove that it is independent of the choice of the basis.

d. For a diagonalizable matrix A give and prove the formula of A^k for $k \in \mathbb{N}^*$.

e. Describe the manner to transform a differential equation of order $n \geq 2$ into a system of first order differential equations.

Exercise 2 Say if the following statements are TRUE or FALSE with a very concise explanation. In case of several errors in a statement give all of them. What matters is **only** the concise explanations.

a. The function $x \to \int_1^{\infty} \frac{1}{x^2 t} dt$ is continuous over $\mathbb{R}^*$.

b. For a 2x2 matrix A, if $\det(A) = 3$ then $\det(2A) = 6$.

c. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

d. A 3x3 matrix can have more than three different eigenvectors.

e. Let $\mathcal{U}$ a subspace of a vector space $\mathcal{V}$ and $P_{\mathcal{U}}$ the orthogonal projection on $\mathcal{U}$, there exists a non vanishing $v \in \mathcal{V}$ such that $P_{\mathcal{U}}(v) = 2v$.

f. For a diagonalizable matrix A, $e^{A^T} = (e^A)^T$ where A^T is the transpose of A.

g. The solution $t \to y(t)$ of the Cauchy problem $y'(t) = \cos(t)y(t) + \ln((t-1)^2)$ for $t \geq 0$ and $y(0) = y_0$ is given by

$$y(t) = e^{\cos(t)t} y_0 + \int_0^t e^{\cos(t)(t-s)} \ln((s-1)^2)\ ds.$$

Exercise 3 Let the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 3 & 1 \\ 1 & -1 & 1 \end{pmatrix}$.

a. With a minimum of computation prove that A is diagonalizable.

b. Calculate the general solution to the differential system $\mathbf{y}'(t) = A\mathbf{y}(t)$.

c. Calculate the first column of A^{-1}.

d. Deduce the solution to the Cauchy problem $\mathbf{y}'(t) = A\mathbf{y}(t) + \mathbf{b}$ with $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{y}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

e. Propose an alternate method of calculation of the solution to the Cauchy problem and justify that its use is allowed in this case. As its computation may be too long it is not demanded, however the precise definition of its components is required.

Exercise 4 For $n \in \mathbb{N}^*$, let $M_n(\mathbb{R})$ the $\mathbb{R}$-vector space of $n \times n$ matrices with real coefficients. Let us admit that the mapping

$$\varphi : \begin{array}{ccc} M_n(\mathbb{R}) \times M_n(\mathbb{R}) & \to & \mathbb{R} \\ (A, B) & \mapsto & tr(A^T B) \end{array}$$

is an inner product and we denote by $A \mapsto ||A||$ the associated norm.

a. For two matrices A, $B \in M_n(\mathbb{R})$ give the formulas of $||A||$, $||B||$ and of the inner product $\varphi(A, B)$ with respect to the matrix coefficients a_{ij} and b_{ij}.

b. For $n = 2$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ calculate the norms $||A||$, $||B||$ and the inner product $\varphi(A, B)$.

c. Find an orthonormal basis of the subspace $U = span\{A, B\}$ of $M_2(\mathbb{R})$ generated by A and B.

d. Find the matrix $D \in U$ which is the closest from $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for the distance defined by the norm $A \to ||A||$.

Exercise 5 We use the notations of Exercise 5. Let $f_m(A) = \sum_{k=0}^{m} \dfrac{A^k}{k!}$ a function defined for any $A \in M_n(\mathbb{R})$ to $M_n(\mathbb{R})$.

a. Propose a definition of the convergence of a series $\sum_{k=0}^{\infty} A_k$ of matrices $A_k \in M_n(\mathbb{R})$.

b. For any matrices A and B in $M_n(\mathbb{R})$, express the norm $||AB||$ with respect to their coefficients a_{ij} and b_{ij}.

c. Using the Cauchy-Schwarz inequality for the inner product $(u, v) = \sum_{k=1}^{n} u_k v_k$ of two vectors $u, v \in \mathbb{R}^n$ prove the inequality $||AB|| \leq ||A|| \times ||B||$.

d. Deduce that $||f_m(A)||$ is bounded and conclude to the existence of the limit $\lim_{m\to\infty} ||f_m(A)||$.

Remark: This yields another kind of convergence of the series of matrices that is used to justify the definition of the exponential of a matrix

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

for any $A \in M_n(\mathbb{R})$.