SQ 28, Spring 2008 - QUIZ 3 -

Friday 27th of June, 8:00-10:00

Exercise 1 a. What is a random variable?
b. What is a sample of random variables?
c. What is a test? Describe its main elements?

Exercise 2 Let X_i, $i = 1, .., 10$ be independent random variables, each uniformly distributed over $(0, 1)$. Calculate an approximation to $P(\sum_{i=1}^{10} X_i > 6)$ based on the central limit theorem.

Exercise 3 Time between two successive breakdowns of an electronic system is a random variable X with an exponential p.d.f.

$$f(x) = \frac{1}{\theta} e^{-x/\theta} \text{ for } x > 0 \text{ and } 0 \text{ otherwise.}$$

a. First, consider Y a random variable with a Gamma distribution of parameters (λ, m), that we denote $Y \sim \Gamma(\lambda, m)$, with p.d.f.

$$\begin{aligned} f_m(x) &= e^{-\lambda x} \frac{\lambda^m x^{m-1}}{(m-1)!} \text{ for } x > 0 \\ &= 0 \text{ otherwise} \end{aligned}$$

where $\lambda > 0$ and $m \in \mathbb{N}$. Prove that the random variable $\lambda Y \sim \Gamma(1, m)$.

b. Prove that the generating function of Y is $\psi(t) = (\lambda/(\lambda - t))^m$. *Hint: Use the fact that* f_m *is a p.d.f..*

c. Use this result to prove that the expected time between two breakdowns and the standard deviation are equal to θ.

d. Consider a sample of electronic devices, their times between two breakdowns are some r.v. X_i. Prove that $X_1 + X_2 \sim \Gamma(1/\theta, 2)$. Then, prove by induction that $\sum_{i=1}^{n} X_i \sim \Gamma(1/\theta, n)$.

e. Let $a\overline{X}$ be an estimator of θ. Find the constant a so that the mean square error is minimum.

f. Build a confidence interval for the mean time between two breakdowns with a specified confidence coefficient α $(0 < \alpha < 1)$ based on the probability distribution $\Gamma(1, n)$. *Hint: Use the probability inteval* $P(c_1 < \frac{1}{\theta} \sum_{i=1}^{n} X_i < c_2) = \alpha$.

g. Apply the Neyman-Pearson likelihood ratio test theory to find the best α-size test of $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$. In particular, you will show that the critical region is based on the statistic $\frac{1}{\theta_0} \sum_{i=1}^{n} X_i$.

h. Test $H_0 : \theta = 1$ against the alternative $H_1 : \theta \neq 1$, for $n = 10$, $\overline{x} = 1.5$ and for a signification level $\alpha = 0.05$. For $T \sim \Gamma(1, 10)$, $P(T > 15.7) = 0.05$.

Exercise 4 Heart rate reduction A new drug for inducing a temporary reduction in a patient's heart rate is to be compared with a standard drug. The drugs are to be administrated to a patient at rest, and the percentage reduction in the heart rate after 5 minutes is to be measured. Since the drug efficacy is expected to depend heavily on the particular patient involved, a paired experiment is run whereby each of 40 patients is administrated one drug on one day and the other drug on the following day.

a. Consider paired samples with data observations $(x_1, y_1), .., (x_n, y_n)$ corresponding to the heart rate of the n patients for the two days. Based on the differences $z_i = x_i - y_i$ for $i = 1, .., n$, use the Z-test to build a test for mean comparison.

b. Apply this test to the Heart rate reduction problem where we found a sample average $\overline{z} = -2.655$ and where we know the standard deviation $\sigma = 3.730$. For $Z \sim N(0, 1)$, $P(Z > 1.65) = 0.95$.

c. In case where the standard deviation is unknown, and the sample standard deviation is $s = 3.730$, apply the t-test. For $T \sim$Student(39), $P(T > 2,02) = 0.95$.

Exercice 5 Let $X_1, .., X_n$ be a sample from the *Weibull* distribution with individual densities

$$f(x, \theta) = \theta x^{\theta - 1} e^{-x^\theta} \text{ for } x > 0$$

where $\theta > 0$ is unknown.

a. Find the likelihood equation, and show that if the likelihood equation has a solution, it must be the MLE. Its explicit calculation is not required (and not possible)!

b. Write the (generalized) likelihood ratio test for the null hypothesis $H_0 : \theta = 1$ versus the alternative $H_1 : \theta \neq 1$.