FINAL EXAM

An electronic calculator, the distributions tables as well as a sheet of personnal notes are allowed for the exam. Any kind of dictionary is permitted

Exercise 1

A same exam is graded twice by two different teachers, A and B. The random variables X and Y correspond to the grades given by respectively A and B. We suppose X and Y follow normal distributions : $X \sim \mathcal{N}(\mu_1, \sigma_1)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2)$. The parameters μ_1, μ_2, σ_1 and σ_2 are unknown. A random sample of size $n = 21$ gave the following point estimates for the means and the standard deviations :

- *Teacher A, $\overline{x}_1 = 8.7$; $s_1 = 1.6$*
- *Teacher B, $\overline{x}_2 = 9.4$; $s_2 = 2.2$*

We would like to know if the two teachers grade differently.

1. *Can we accept at a level of signifiance of 5% the equality of variances ?*

2. *We now assume $\sigma^2 = \sigma_1^2 = \sigma_2^2$, give a point estimate of σ^2.*

3. *Should we believe the two teachers grade differently (level of significance 5%) ?*

Exercise 2

Let X be the standard normal distribution $X \sim \mathcal{N}(0, 1)$. We denote by $(X_1, \ldots, X_n)$ a random sample of n independent identically distributed random variables with $X_i \sim \mathcal{N}(0, 1)$. We recall that the distribution χ_n^2 (chi square distribution with n degrees of freedom) is defined as follow : $\chi_n^2 = X_1^2 + X_2^2 + \cdots + X_n^2$.

1. *Calculations of $E(\chi_n^2)$ and $V(\chi_n^2)$:*

- **a.** *Without calculation prove that $E(X^2) = 1$ (you may use $V(X)$).*
- **b.** *Show that $E(\chi_n^2) = n$.*
- **c.** *Prove $E(X^4) = 3E(X^2)$ and calculate $V(X^2)$.*
- **d.** *Show that $V(\chi_n^2) = 2n$.*

2. *Let $n = 200$, determine the values α_1 and α_2 so that $P(\chi_{200}^2 \leqslant \alpha_1) = 0.975$ and $P(\chi_{200}^2 \leqslant \alpha_2) = 0.025$*

3. *The standard deviation of the lifetimes of a sample of 200 electric light bulbs was computed to be 100 hours. We suppose the mean is known and the lifetime of the electric ligh bulbs follows a normal distribution. Find a 95% confidence interval for the variance of all such electric light bulbs.*

4. *We now would like to test for another kind of electric light bulbs if the standard deviation is equal to 100 hours. For this new kind of electric light bulbs we have a random sample of 12 electric light bulbs whose lifetimes (in hours) were recorded in the following :*

1200	1100	850	1050	750	1150	950	950	1250	1000	1000	850

- **a.** *Give a point estimate of the mean lifetime and standard deviation for this new kind of electric light bulbs.*
- **b.** *Perfom a test of hypothesis to decide if the standard deviation can be believed to be equal to 100 hours with a level of significance of 1%.*

Exercise 3

In this exercise R denotes a non-negative real number. We consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin [0, R] \\ \dfrac{2x}{R^2} & \text{if } x \in [0, R] \end{cases}$$

1. *Show that f is a density function. We denote by X the random variable with density function f and by F the cumulative distribution function.*

2. *Determine $F(x)$ for $x \in [0, R]$.*

3. *Prove that $E(X) = \dfrac{2R}{3}$ and $V(X) = \dfrac{R^2}{18}$.*

We now suppose $(X_1, \ldots, X_n)$ is a random sample of independent indentically distributed random variables following the same distribution as X. We would like to estimate R.

4. *Let $T_n = \dfrac{3}{2n} \sum_{i=1}^{n} X_i$. Prove that T_n is an unbiased estimator of R. Is T_n convergent ?*

5. *We assume that X corresponds to the temperature at which a certain chemical reaction occurs (the probability of occurences increases as X get closer but less than R and vanishes for $X \geqslant R$. Thus the value R can be understood as the temperature of a change of phase).*

a. *A random sample of 40 observations $(x_1, \ldots, x_{40})$ yielded an estimated mean $\overline{x} = 74$°C. Give a point estimate of R.*

b. *At a level of significance of 5% we would like to test the hypothesis $H_0 : R = 100$°C versus $H_1 : R > 100$°C.*

i. Starting with T_n, construct a test variable Z.

ii. How can you approximate Z ?

iii. Perform the test and give your decision.