FINAL EXAM

An electronic calculator, the distributions tables as well as a sheet of personal notes are allowed for the exam. Any kind of dictionary is permitted

Exercice 1 (7 points)

The goal of this exercise is to answer the following question : if we choose randomly two points on a segment of length 1 what is the probability p that the three pieces formed could be the sides of a triangle ?

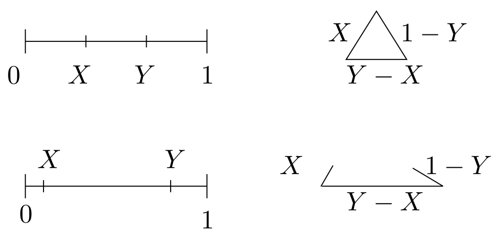

FIGURE 1 – Two ways of cutting the segment in three pieces : the first case gives a triangle, the second does not.

We denote by X the position on $[0, 1]$ of the first point chosen randomly and Y the position on $[0, 1]$ of the second point. We assume the choice of the first and second point are independent and there are no preferred position on the segment (every point could be chosen).

1. Which usual distribution does correspond to the random variables X and Y ? Give the density function $f_X(x)$ of X and $g_Y(y)$ of Y.

2. Give the density function of the joint random variable (X, Y).

3. We recall that three segments of length α, β, γ can be the sides of a triangle if and only if the three following conditions hold :

$$\bullet\ \alpha \leqslant \beta + \gamma \quad \bullet\ \beta \leqslant \alpha + \gamma \quad \bullet\ \gamma \leqslant \alpha + \beta$$

- **a.** if we assume $X \leqslant Y$ give the three conditions in terms of X and Y which will insure that the segments can be the sides of a triangle.
- **b.** Same question as before if we assume $Y \leqslant X$.
- **c.** Draw in the (x, y) plane the domain corresponding to the event "the three segments will be sides of a triangles" (you need to consider both cases $X \leqslant Y$ and $Y \leqslant X$).
- **d.** Conclude $p = \frac{1}{4}$

4. We change our procedure to cut the segment in three pieces : we choose the first point $X \in [0, 1]$, then we choose independently $Z \in [0, 1]$ and decide to take the second point on $Y = ZX$ in $[0, X]$.

- **a.** Prove that this procedure gives three segments of length ZX, $X(1 - Z)$ and $1 - X$.
- **b.** Draw in the (x, z) the the domain corresponding to "the three segments will be sides of a triangle".
- **c.** Calculate the area of this domain and give the probability of obtaining a triangle with that new procedure.

Exercice 2 (6 points)

A sample of 10 measurements of the diameter of a sphere gave a mean $\overline{x} = 4.38$ inches and a standard deviation $s = 0.06$ inch. We assume the ramdom variable which represents the measurements of the diameter follows a normal distribution $\mathcal{N}(\mu, \sigma)$.

1. Give a point estimate of μ and σ^2.

2. Find a 95% confidence interval for the mean μ.

3. Find a 80% confidence interval for the variance σ^2.

4. **a.** Let us now consider a sample of size n with $n > 50$. Find in terms of n the length of the confidence interval of μ with a level a confidence of 95%.

b. We want the length of this inteval to be smaller than 0.02 inches. What should be approximately the size of the sample ? (you can use as a point estimate of the variance the value calculated in **1**).

Exercice 3 (7 points)

The following exercise takes place in 2020. The teachers of SQ20 and SQ28 may have changed (or not) :

1. After the final exam of SQ20 the teacher grades a sample of 10 papers in order to know the quality of the students in 2020 before eventually changing his grading system. We assume the results on the sample are

12	10	7	4	15	13	11	17	10	12

a. Give a point estimate of the mean and standard deviation of the popultation.

b. The teacher would like to know if the student are really good this year (or maybe his course was clear). For that reason he decides to perform a test on hypothesis to discuss $\mu = 9$ (the usual mean) versus $\mu > 9$. Perform such a test (you will describe the hypothesis H_0 and H_1, find the test variable, compute the domain of acceptance and conclude).

c. Compute the risk β if we test $H_0 : \mu = 9$ versus $H_1 : \mu = 11$ (you'll show $0.5 \leqslant \beta \leqslant 0.55$).

2. In SQ28 in order to test the quality of the papers the teacher plays Head and Tails 40 times. The number of Heads divided by two gives the mean that the teacher will get by adapting his grading system. The coin used to calculate the mean has a probability p to turn Head.

a. We denote by X the random variable which will give the mean of the final exam. What classical distribution corresponds to $2X$.

b. Calculate $E(2X)$ and $V(2X)$ and give a normal approximation of $2X$ by the Central Limit Theorem.

c. The mean at the final exam is 12. Perform a test on hypothesis at the level of significance of 95% to decide if the coin is well balanced (under the hypothesis $H_0 : p = \frac{1}{2}$ you can use the test variable $Z = \dfrac{2X - 40}{\sqrt{\frac{1}{160}}}$ which can be approximated by the standard normal distribution).