Probability and Statistics - SQ28
Tronc commun
Final - Spring 2014

Test duration : 2 hours

Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification will not receive full credit.

An A4 sheet with handwritten notes is allowed, as is the table of the standard normal distribution.
Calculators are allowed.

Exercise 1 (6 points)

A company produces corks[1] and uses chemicals to speed up the growth of its trees. Those treatments can sicken the trees and then affect the taste of the bottles of wine: it gives a cork taint to the wine. In the following, we will denote by p the proportion of corks having this default.
A group of winemakers gather and tastes 215 bottles, 13 of which have a cork taint.

1. Give a point estimate of p.
2. Build a 99%-confidence interval for p.
3. The company says that 3% of its corks have a default. Test its claim with a risk level of 1%.

Exercise 2 (6 points)

On her computer, Alice has a function that generates $\mathcal{N}(0,1)$-distributed random numbers. Bob tweaked it so that it will generate $\mathcal{N}(\mu,\sigma^2)$ random numbers. Bob challenged Alice to find μ and σ^2.

1. Alice ran the function 11 times. Here is the output:

$$1.71 * 0.89 * 0.43 * 1.58 * 1.74 * (-0.79) * 1.87 * 0.02 * (-0.04) * 0.22 * (-0.19)$$

   (a) Give a point estimate of μ.
   (b) Give a point estimate of σ^2.
   (c) Build a 95%-confidence interval for μ.
   (d) Build a 99%-confidence interval for σ^2.

2. In your opinion, how did Bob procede to generate $\mathcal{N}(\mu,\sigma^2)$ random numbers, having at his disposal the original function that generates $\mathcal{N}(0,1)$ numbers?

[1] cork=bouchon

Exercise 3 (8 points)

Suppose a machine has a certain probability of breaking down. Denote by:

- X the time (in years) between the start-up of the machine and its first breakdown.
- Y the time (in years) between the first and the second breakdown.

After two breakdowns, the machine is tossed away. We assume that X and Y are two independent, exponentially-distributed variables with the same parameter $\lambda > 0$.

1. (a) Give a density f of X and the cumulative distribution F of X.
   (b) What is the average time span between two breakdowns?
2. What is the probability that the first breakdown will occur after more than 3 years and that the second breakdown will occur more than 3 years after the first one?

   breakdowns will occur after more than 3 years?
3. Let S be the total lifespan of the machine: $S = X + Y$.
   (a) Compute $\mathbb{E}(S)$ and $\mathbb{V}\mathrm{ar}(S)$.
   (b) Let h be the probability density function of S. Show that:
   $$h(t) = \begin{cases} \lambda^2 \, t \, \mathrm{e}^{-\lambda t} & \text{if } t \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$
   (c) Compute $\mathbb{P}(S > x)$ for all $x > 0$.
4. In this question, we assume that the average lifetime of the machine is 4 years. Compute the probability that it will run more than 6 years.