PROBABILITY AND STATISTICS - SQ28
TRONC COMMUN
FINAL - SPRING 2015

TEST DURATION : 2 HOURS

Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification will not receive full credit.

Calculators are allowed.

Please write the solutions of the exercises on different sheets.

Exercise 1. A quality control service wants to test the capacity of some bottles. The results of eighty-one measurements made by the service are shown in the following table:

Volume (in ml)	57,95	57,99	58,03	58,07	58,11
Number of bottles	3	10	39	21	8

Denote by X the random variable representing the volume of a bottle.

1. Give a point estimate of $\overline{X}$ and $\mathbb{V}\mathrm{ar}(X)$.

The company selling the bottles advertises that the volume of its bottles is 58 ml. To test this claim, we will assume that X is normally distributed with unknown expected value μ and variance σ^2. We will write:

$$\mathcal{H}_0 \ : \ \mu = 58, \qquad \mathcal{H}_1 \ : \ \mu \neq 58.$$

2. Denote by $(X_1, \dots, X_n)$ a random sample of X. Let

$$Z = \sqrt{n} \cdot \frac{\overline{X}_n - \mu}{S_n},$$

where

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2.$$

Assume that $\mathcal{H}_0$ is true.

(a) What is the probability distribution of Z?

(b) Find $t > 0$ such that $\mathbb{P}(-t \leqslant Z \leqslant t) = 95\%$.

(c) Find an interval I_0 such that $\mathbb{P}(\overline{X} \in I_0) = 95\%$.

3. Given the results of the eighty-one measurements, will the quality control service accept $\mathcal{H}_0$? Explain.

Don't forget to take a new sheet for exercise 2.

Exercise 2. Let $\lambda > 0$ and f be the function:

$$f : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} e^{\lambda - t} & \text{if } t \geqslant \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

1. (a) For all $x \geqslant \lambda$, compute $\displaystyle\int_{\lambda}^{x} f(t)\mathrm{d}t$.
   (b) Show that f is a probability density function.

Let X be a random variable with probability density function f.

2. What is the cumulative distribution function of X?

3. Let $U = X - \lambda$.

   (a) Show that the cumulative distribution function G of U satisfies:

   $$G(x) = \begin{cases} 1 - e^{-x} & \text{if } x \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Deduce the probability distribution of U. What are its expected value and variance?
   (c) Deduce $\mathbb{E}(X)$ and $\mathbb{V}\mathrm{ar}(X)$.

Let $X_1, \dots, X_n$ be n mutually independent random variables with probability density function f. We want to estimate the parameter λ. Let:

$$S_n = \overline{X}_n - 1 = \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - 1 \quad \text{and} \quad Y_n = \min(X_1, X_2, \dots, X_n).$$

4. (a) Show that S_n is an unbiased estimator of λ.
   (b) Show that S_n is a consistent estimator of λ.

5. Show that the cumulative distribution function F_n of Y_n is given by:

   $$F_n(x) = \begin{cases} 1 - e^{n(\lambda - x)} & \text{if } x \geqslant \lambda, \\ 0 & \text{if } x < \lambda. \end{cases}$$

6. Let $Z_n = Y_n - \lambda$.

   (a) What is the cumulative distribution function G_n of Z_n?
   (b) Show that Z_n is exponentially distributed and give the corresponding parameter.
   (c) Deduce $\mathbb{E}(Y_n)$ and $\mathbb{V}\mathrm{ar}(Y_n)$.

7. Let $T_n = Y_n - \dfrac{1}{n}$.

   (a) Show that T_n is a consistent and unbiased estimator of λ.
   (b) Is S_n a better estimator than T_n?