

SQ28 - Spring 2017

Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. **Correct answers without proper justification will not receive any credit.**

Exercise 1 (Benoît and the mosquito). Benoît has a hard time falling asleep when there is a mosquito in his bedroom as it will fly near his ears from time to time. Denote by X a random variable that represents the time Benoît needs to fall asleep and by Y the time before a mosquito attack. Assume that:

$$X \sim \mathcal{E}(\lambda) \quad \text{and} \quad Y \sim \mathcal{E}(\mu),$$

where $\lambda, \mu \in \mathbb{R}_+$. Assume also that X and Y are independent.

1. Show that the exponential distribution is memoryless, i.e. for all $s, t > 0$:

$$\mathbb{P}_{(X>s)}(X > s + t) = \mathbb{P}(X > t)$$

2. Does the model seem legitimate?

3. One can show that:

$$\mathbb{P}(X < Y) = \int_0^{+\infty} \mathbb{P}(X < y) \mu e^{-\mu y} dy.$$

- (a) Compute $\mathbb{P}(X < y)$ for all $y \in \mathbb{R}_+$.

- (b) Deduce that:

$$\mathbb{P}(X < Y) = \frac{\lambda}{\lambda + \mu}$$

- (c) Give a similar result for $\mathbb{P}(Y < X)$.

- (d) Deduce $\mathbb{P}(X = Y)$.

4. From now on, we will assume that there are 20 mosquitos in Benoît's bedroom and that the times of the attacks are still exponentially distributed with parameter μ . We will also assume that the attacks are mutually independent and that Benoît has ninja skills: he kills each mosquito after its first attack. The aim of the following questions is to give a confidence interval for μ . Denote by (X_1, \dots, X_{20}) the times of the attacks.

- (a) One can show that if (X_1, \dots, X_n) is a random sample of $\mathcal{E}(\mu)$ then:

$$\mu \times \left(\sum_{i=1}^n X_i \right) \sim \text{Er}(n),$$

where Er denotes the Erlang distribution. Find α and β in the Erlang table¹ such that:

$$\mathbb{P}(\alpha \leq \text{Er}(20) \leq \beta) = 95\%.$$

- (b) Deduce a confidence interval at a level of 95% for μ .
- (c) As Benoît couldn't sleep last night with his 20 mosquitos, he took the time to write down the times of the attacks. The empirical mean he observed was $\bar{x} = 5.2$ minutes. Give the corresponding observed confidence interval.

Exercise 2 (Scilab). In this exercise, you will write a program to estimate the probability $\mathbb{P}(X < Y)$ from exercise 1. This exercise is independent from exercise 1. If you don't remember the syntax of a command, just write it in plain english.

1. Write a Scilab program that takes two parameters λ and μ and simulates X and Y where

$$X \sim \mathcal{E}(\lambda) \quad \text{and} \quad Y \sim \mathcal{E}(\mu)$$

Your program will output 1 if $X < Y$ and 0 otherwise.

2. Write a program that gives an estimation of $\mathbb{P}(X < Y)$: on what result from the course is this estimation based?

¹the Erlang table works exactly like the one of the χ^2 distribution, you will find it on page 2.

Exercise 3. Damien has a deck of 32 cards. He decides to draw 5 cards at once (randomly).

1. What is the probability that he will get exactly 2 queens and one king?
2. What is the probability that he will get at least one king?

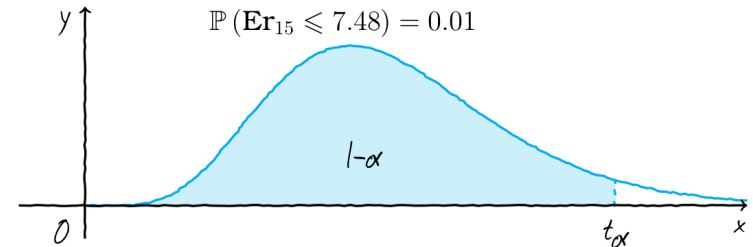
Exercise 4. Let $a \in \mathbb{R}$ and let f be the function:

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto \begin{cases} a & \text{if } 0 \leq x \leq 1, \\ 2a & \text{if } 1 < x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

1. Plot f .
2. Find a such that f is a probability density function.
3. Let X be a continuous random variable with pdf f .
 - (a) Compute $\mathbb{P}(X \leq 1)$.
 - (b) Compute $\mathbb{E}(X)$.
 - (c) Compute the cumulative distribution function of X .

Erlang table



k	0.005	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99	0.995
1	0.01	0.01	0.03	0.05	0.11	2.3	3.0	3.69	4.61	5.3
2	0.1	0.15	0.24	0.36	0.53	3.89	4.74	5.57	6.64	7.43
3	0.34	0.44	0.62	0.82	1.1	5.32	6.3	7.22	8.41	9.27
4	0.67	0.82	1.09	1.37	1.74	6.68	7.75	8.77	10.05	10.98
5	1.08	1.28	1.62	1.97	2.43	7.99	9.15	10.24	11.6	12.59
6	1.54	1.79	2.2	2.61	3.15	9.27	10.51	11.67	13.11	14.15
7	2.04	2.33	2.81	3.29	3.89	10.53	11.84	13.06	14.57	15.66
8	2.57	2.91	3.45	3.98	4.66	11.77	13.15	14.42	16.0	17.13
9	3.13	3.51	4.12	4.7	5.43	12.99	14.43	15.76	17.4	18.58
10	3.72	4.13	4.8	5.43	6.22	14.21	15.71	17.08	18.78	20.0
11	4.32	4.77	5.49	6.17	7.02	15.41	16.96	18.39	20.14	21.4
12	4.94	5.43	6.2	6.92	7.83	16.6	18.21	19.68	21.49	22.78
13	5.58	6.1	6.92	7.69	8.65	17.78	19.44	20.96	22.82	24.14
14	6.23	6.78	7.65	8.46	9.47	18.96	20.67	22.23	24.14	25.5
15	6.89	7.48	8.4	9.25	10.3	20.13	21.89	23.49	25.45	26.84
16	7.57	8.18	9.15	10.04	11.14	21.29	23.1	24.74	26.74	28.16
17	8.25	8.89	9.9	10.83	11.98	22.45	24.3	25.98	28.03	29.48
18	8.94	9.62	10.67	11.63	12.82	23.61	25.5	27.22	29.31	30.79
19	9.64	10.35	11.44	12.44	13.67	24.76	26.69	28.45	30.58	32.09
20	10.35	11.08	12.22	13.25	14.53	25.9	27.88	29.67	31.85	33.38
21	11.07	11.83	13.0	14.07	15.38	27.05	29.06	30.89	33.1	34.67
22	11.79	12.57	13.79	14.89	16.24	28.18	30.24	32.1	34.35	35.95
23	12.52	13.33	14.58	15.72	17.11	29.32	31.41	33.31	35.6	37.22
24	13.26	14.09	15.38	16.55	17.97	30.45	32.59	34.51	36.84	38.48
25	14.0	14.85	16.18	17.38	18.84	31.58	33.75	35.71	38.08	39.74
26	14.74	15.62	16.98	18.22	19.72	32.71	34.92	36.9	39.31	41.0
27	15.49	16.4	17.79	19.06	20.59	33.84	36.08	38.1	40.53	42.25
28	16.25	17.17	18.61	19.9	21.47	34.96	37.23	39.28	41.76	43.5
29	17.0	17.96	19.42	20.75	22.35	36.08	38.39	40.47	42.98	44.74
30	17.77	18.74	20.24	21.59	23.23	37.2	39.54	41.65	44.19	45.98
40	25.59	26.77	28.58	30.2	32.14	48.29	50.94	53.31	56.16	58.16
50	33.66	35.03	37.11	38.96	41.18	59.25	62.17	64.78	67.9	70.08
60	41.93	43.46	45.79	47.85	50.31	70.12	73.28	76.11	79.48	81.82
70	50.33	52.02	54.57	56.83	59.51	80.91	84.31	87.32	90.92	93.42
80	58.84	60.67	63.44	65.88	68.77	91.66	95.26	98.46	102.27	104.91
90	67.44	69.41	72.37	74.98	78.08	102.35	106.15	109.52	113.53	116.31
100	76.12	78.22	81.36	84.14	87.42	113.01	117.0	120.53	124.72	127.63