SQ28 - Spring 2018

Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. ***Correct answers without proper justification will not receive any credit.***

Exercise 1. Questions 1. and 2. are independent. Let $U \sim \mathcal{U}([0,1])$ and $V = 1 - U$. Denote by F_U and F_V the cumulative distribution functions of U and V respectively.

1. (a) Show that for all $t \in \mathbb{R}$:

$$F_U(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leqslant t < 1 \\ 1 & \text{if } t \geqslant 1. \end{cases}$$

   (b) Using the fact that:

$$\forall t \in \mathbb{R}, \quad \mathbb{P}(V \leqslant t) = \mathbb{P}(1 - U \leqslant t),$$

   deduce F_V.

   (c) What is the probability distribution of V?

2. (a) Compute $\mathbb{C}\mathrm{ov}(U, V)$.

   (b) Are U and V independent?

Exercise 2. Let $X \sim \mathcal{E}(\lambda)$ where $\lambda > 0$, and let $Y = \lfloor X \rfloor + 1$.

1. Show that for all $n \in \mathbb{N}$:

$$\mathbb{P}(n \leqslant X < n+1) = (1 - p_\lambda)^n p_\lambda,$$

   where $p_\lambda \in]0, 1[$ does not depend on n.

2. Deduce the probability distribution of Y.
   Hint: $\lfloor X \rfloor = n \iff n \leqslant X < n + 1$

Exercise 3. Let $X_1, \ldots, X_n$ be a random sample of the $\mathcal{U}([0,1])$ distribution. Let:

$$Y = \sum_{k=1}^{n} X_k.$$

The probability distribution of Y is called the Irwin-Hall distribution with parameter n. We will write $Y \sim IH_n$.

1. Let:

$$Z = \sqrt{n} \cdot \frac{\overline{X}_n - 1/2}{\sqrt{1/12}}.$$

   (a) Show that the distribution of Z can be approximated with an $\mathcal{N}(0,1)$.

   (b) Give a normal approximation of $\mathbb{P}(5 \leqslant Y \leqslant 10)$ for $n = 15$.

2. A computer receives a number n of tasks during the day. It processes task number k in U_k minutes, independently from the others, where U_k has the $\mathcal{U}([0,1])$ distribution. Denote by Y the total processing time in one day.

   (a) Assume that $n = 20$. Using the Irwin-Hall table, find $(u, v) \in \mathbb{R}^2$ such that:

$$\mathbb{P}(u \leqslant Y \leqslant v) = 99\%.$$

   (b) On a given day, the computer worked for $y = 8.28$ minutes.

      (i) Test whether the number of tasks that day was equal to $n = 20$ or not using a risk level of 1%.

      (ii) In order to reject $\mathcal{H}_0 : n = 20$, what risk level would one have to take?

Exercise 4 (Scilab)**.** The aim of this exercise is to simulate Y from exercise 3 and compare the distribution of Z to the one of the $\mathcal{N}(0,1)$. This exercise is independent from exercise 3.

1. Write a scilab function that takes as an argument n and outputs 1 realization of the Irwin-Hall distribution (i.e. a sum of n independent $\mathcal{U}([0,1])$ variables).

2. Write a scilab function that takes as an argument n and outputs 1 realization of Z from exercise 3.

3. Write a scilab function that takes as an argument N and n which will:

    - plot a histogram of N realizations of Z from exercise 3 with parameter n,
    - plot the pdf of the $\mathcal{N}(0,1)$ distribution against it.

Exercise 5. A teacher has 142 exercises under his SQ28 folder on his computer:

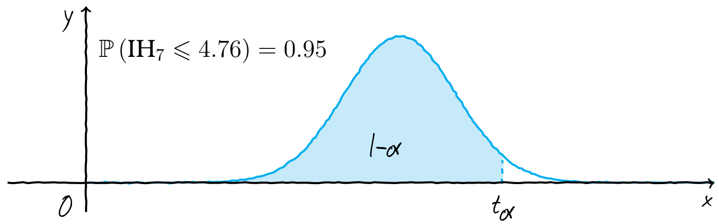
- 28 on combinatorics,
- 22 on discrete random variables,
- 37 on continuous random variables,
- 12 on point estimation,
- 15 on confidence intervals.

If he picks randomly 5 exercises to create a test, ignoring the order of the exercises:

1. how many different tests can he make?

2. what is the probability that his students will get exactly one combinatorics exercise and two exercises on continuous random variables?

Irwin-Hall table



n	0.005	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99	0.995
2	0.1	0.14	0.22	0.32	0.45	1.55	1.68	1.78	1.86	1.9
3	0.31	0.39	0.53	0.67	0.84	2.16	2.33	2.47	2.61	2.69
4	0.59	0.7	0.88	1.05	1.25	2.75	2.95	3.12	3.3	3.41
5	0.9	1.04	1.25	1.43	1.66	3.34	3.57	3.75	3.96	4.1
6	1.24	1.39	1.62	1.83	2.08	3.92	4.17	4.38	4.61	4.76
7	1.59	1.76	2.01	2.24	2.51	4.49	4.76	4.99	5.24	5.41
8	1.95	2.13	2.41	2.65	2.94	5.06	5.35	5.59	5.87	6.05
9	2.32	2.51	2.81	3.07	3.38	5.62	5.93	6.19	6.49	6.68
10	2.7	2.9	3.22	3.5	3.82	6.18	6.5	6.78	7.1	7.3
11	3.07	3.3	3.63	3.92	4.27	6.73	7.08	7.37	7.7	7.93
12	3.47	3.7	4.05	4.35	4.71	7.29	7.65	7.95	8.3	8.53
13	3.86	4.1	4.47	4.79	5.16	7.84	8.21	8.53	8.9	9.14
14	4.25	4.51	4.89	5.22	5.61	8.39	8.78	9.11	9.49	9.75
15	4.66	4.92	5.31	5.66	6.06	8.94	9.34	9.69	10.08	10.34
16	5.06	5.34	5.74	6.1	6.51	9.49	9.9	10.26	10.66	10.94
17	5.46	5.75	6.17	6.54	6.97	10.03	10.46	10.83	11.25	11.54
18	5.87	6.17	6.6	6.98	7.42	10.58	11.02	11.4	11.83	12.13
19	6.29	6.6	7.04	7.43	7.88	11.12	11.57	11.96	12.4	12.71
20	6.7	7.01	7.48	7.88	8.34	11.66	12.12	12.52	12.99	13.3
21	7.12	7.44	7.91	8.32	8.8	12.2	12.68	13.09	13.56	13.88
22	7.54	7.86	8.35	8.77	9.26	12.74	13.23	13.65	14.14	14.46
23	7.96	8.3	8.79	9.22	9.72	13.28	13.78	14.21	14.7	15.04
24	8.39	8.73	9.23	9.67	10.18	13.82	14.33	14.77	15.27	15.61
25	8.81	9.16	9.67	10.13	10.64	14.36	14.87	15.33	15.84	16.19
26	9.24	9.59	10.12	10.58	11.11	14.89	15.42	15.88	16.41	16.76
27	9.66	10.03	10.57	11.03	11.57	15.43	15.97	16.43	16.97	17.34
28	10.09	10.46	11.01	11.49	12.04	15.96	16.51	16.99	17.54	17.91
29	10.52	10.9	11.46	11.94	12.5	16.5	17.06	17.54	18.1	18.48
30	10.96	11.34	11.91	12.4	12.97	17.03	17.6	18.09	18.66	19.04
40	15.31	15.76	16.43	17.0	17.66	22.34	23.0	23.57	24.24	24.69
50	19.75	20.26	21.0	21.64	22.38	27.62	28.36	29.0	29.74	30.25