utbm université de technologie Belfort-Montbéliard

SQ 28 Mid-term exam

Monday April 24th 2006

For this test, you may use an electronic calculator and the Tables of Statistics. Length 2 hours.

We consider a deck of 52 cards with 13 cards of each colour : Spades ♠, Hearts ♥, Diamonds ♦ and Clubs ♣. In each colour, there are 13 cards (values), in descending order, Ace = A, King = K, Queen = Q, Jack = J, 10, .., 3 and 2. These 52 cards are randomly dealt to four players sitting around a table, on the cardinal points : clockwise N(orth) E(ast), S(outh) and W(est).

We call *hand* a set of 13 cards (order does not matter) out of the 52 cards. We call *deal* any sharing between 4 hands for 4 players. Two deals are therefore different if, and only if, at least one player has not got exactly the same hand.

I. Q 1 – Counting

1–1 : What is the number of hands a player can get ?

1–2 : What is the number of deals four players can get ?

1–3 : What is the number of deals in which every player gets 13 cards of the same colour ?

1–4 : What is the number of deals in which a player gets 13 cards with 8 cards of one colour and 5 cards of another one ?

1–5 : What is the probability of this event (previous question). (answer = $3{,}13 \, . \, 10^{-5}$). Let T be the number of deals a player plays until he gets such a hand. What is the law of T ?

1–6 : Assuming that this player plays a tournament of 22 deals, 6 days a week an during a year, how long, on average, must he wait until he gets once more such a hand ?

II. Q 2 – Evaluation of the strength of a hand :

1°) Frenchman Pierre Albarran popularized the following evaluation method for the strength of a hand:

We allocate 4 points for each A(ce), 3 points for each K(ing), 2 for each Q(ueen), 1 for each J(ack) and nothing for the other values.

If N_1 is the number of A in one hand, N_2 the number of K, N_3 the number of Q and N_4 the number of J, the strength in points (called *points of honours*) is a random variable $F = 4\,N_1 + 3\,N_2 + 2\,N_3 + N_4$.

2–1 : What is the strength of a deck of 52 cards ? Give without calculation the expected value of F.

2–2 : What are the possible values of F ?

2–3 : Prove that variables N_1 to N_4 follow the same hypergeometric distribution H(N, n, p) and give their parameters. Calculate $E(N_1)$ and $Var(N_1)$.

2–4 : Then find E(F) again.

2°) 2–5 : We want to calculate Var(F), variables N_1 to N_4 being not independent.

Prove $Var(N_1 + N_2) = Var(N_1) + Var(N_2) + 2\,Covar(N_1, N_2)$ and then that

$$Var(F) = 30\,Var(N_1) + 70\,Covar(N_1, N_2).$$

2–6 : It is about to calculate $Covar(N_1, N_2)$. Build the table of the distribution of the couple (N_1, N_2).

2–7 : Deduce $E(N_1, N_2) = 0.9412$.

2–8 : Then prove that $Var(F) = 17.06$.

2–9 : We admit that F is normal $N(m, \sigma)$ with $m = 10$ and $\sigma = 4.13$. Calculate, with the tables, the probabilities of the following events :

A : The hand has less than 15 points ($F \leq 14.5$)

B : The hand has between 15 and 17 points ($14.5 \le F \le 17.5$).
C : The hand has less than 8 points ($F \le 7.5$)

3°) A student, whose name will be kept secret, follows UV BR 01, and he must therefore play 5 deals and wins at least 3 of them to pass his exam. For each deal the probability to win is 0.34.

2−10 : Let N be the number of deals he wins. What is the distribution of N, its expectation and its variance ? Calculate the probability of the events :

A : “He passes his exam” B = “he passes his exam knowing that he lost the first deal”

For extra points (beyond 20), you may solve Q3 on the English <u>or</u> the French text.

If you are short of time, you may skip the explanations, in order to have enough time left to answer the questions.

III. Q 3 – Sharing out cards between players and game strategy :

Actually, in the game of Bridge, players play two against two (NS against EW) and the *announcer* (i.e. the one who declared the final contract) that we assume to be S, has the advantage to know his partner’s game (here North called the *dummy*) because North’s game is displayed on the table so that everybody can see it at the beginning of the actual game. Here is the situation :

Hand of N :	♠ 6 5 4 3	♥ 6 5	♦ Q J 10 3 2	♣ A K
Hand of S :	♠ A K J 10 2	♥ A 4 3 2	♦ A K	♣ 3 2

1°) Big-time gambler Cora’s strategy :

Cora plays S, and after eight tricks, each player has five cards in his hand. W and E have together 10 cards left : ♠ Q 9 8 and 7, and six other cards of other colours.

Cora (S), whose hand is ♠ A K J 10 and 2, looses the deal if one of her opponents (W or E) has Q ♠ and at least two others ♠.

a) What is the number of distributions of those ten cards between W and E ?
b) Calculate the probability that one opponent has Q of ♠ and at least two other ♠.
c) Deduce the probability that Cora wins.

2°) Cora plays with an extra information:

a) Assume we know W had seven ♥ at the beginning of the game. Let R be the number of ♥ he has left after the 8th trick. What are the possible values of R ? Calculate $p(R = k)$ for those possible values.
b) Calculate the probabilities $p(\text{Cora wins} \mid R = k)$ for $k \in \{0, 1, 2, 3, 4, 5\}$, and p(Cora wins).

Vocabulary :

trick = levée ou pli (= les quatre cartes ramassées par un joueur après que chacun des 4 ait joué une carte), deal = donne = au bridge distribution de 13 cartes à chacun des quatre joueurs, deck of cards = jeu de cartes , the dummy = le mort , or = ou

BR 01 = Bridge 01 = UV de culture générale qui n’est pas (du moins pas encore) assurée à L’UTBM. Pour une ouverture possible au prochain semestre, se renseigner auprès de M. Claude Petitjean.


Paul CÉZANNE : Les joueurs de cartes

Si on ne peut plus tricher aux cartes avec ses amis, ce n’est plus la peine de jouer aux cartes.
Marcel PAGNOL (FRANCE 1895 – 1974)

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Q1 : Counting

1-1 : One hand = 13 cards/52 $\Rightarrow$ number of hands = $\underline{C_{52}^{13} \simeq 6{,}35 \, . \, 10^{11}}$

1-2 one deal = ——— (1^{st} player) and then 13 cards/39 and then 13 cards/26 : nb of deals :
$\underline{C_{52}^{13} \times C_{39}^{13} \times C_{26}^{13} \simeq 5{,}36 . 10^{28}}$

1-3 : same colour : 4 colours $\rightarrow$ 4 players $\Rightarrow$ $\underline{nb = 4! = 24}$

1-4 : 13 cards 8 1^{st} colour 5 2^{nd} colour.
choices : 1^{st} colour/4 $\rightarrow$ 4 8 cards/13 $\rightarrow C_{13}^{8}$ 2^{nd} colour/3 $\rightarrow$ 3 5 cards/13 $\rightarrow C_{13}^{5} \Rightarrow 4 \times C_{13}^{8} \times 3 \times C_{13}^{5}$
$\underline{\text{nb of hands } 8+5 \simeq 1{,}99 \, . \, 10^{7}}$

1-5 $p(8+5) = \dfrac{1{,}99 . 10^{7}}{6{,}35 . 10^{11}} = \underline{3{,}13 \, . \, 10^{-5} = p(8+5)} = p$ (we suppose that the deck is properly mixed before dealing)

T = nb of deals until a (8+5) hand.
$\uparrow$ independent

$$\begin{cases} p & 8+5 \\ 1-p & \overline{8+5} \end{cases} \Rightarrow T \sim \mathcal{G}(p)$$

1-6 : Since $E(T) = \dfrac{1}{p}$, a player has, on average, to wait $\dfrac{1}{3{,}13 . 10^{-5}}$ deals before he gets such a hand.
nb of deals to wait = 31 949 = $\underline{13{,}39 \text{ years}}$ = 13 years 142 days

Q-2 : Strength of a hand :

1°) 2-1 : one deck of cards = 4A, 4K, 4Q, 4J $\Rightarrow$ total strength = $4 \times (4+3+2+1)$ = $\underline{40 \text{ points} = \text{total strength}}$

40 points are to be shared between 4 players $\Rightarrow$ expected value for each player $\underline{E(F) = 10}$

2-2 : Possible values of F : lower bound = no honours $\rightarrow$ F = 0
upper bound = a hand with 4A, 4K, 4Q and 1J $\rightarrow$ F = 37 $\Big\} \Rightarrow F(\Omega) = \{1, \dots, 37\}$

2-3 : N_1 = number of aces in one hand, the cards being simultaneously drawn from the deck,
= ——— /13 cards $\quad N_1 \sim \mathcal{H}\left(N = 52, n = 13, p = \frac{4}{52} = \frac{1}{13}\right)$

Therefore $\underline{E(N_1) = np = 1}$, $Var(N_1) = np(1-p) \dfrac{N-n}{N-1} = \dfrac{12}{17} \simeq \underline{0{,}7 = Var(N_1)}$

We have (obviously) the same reasoning for N_2, N_3 and N_4.

2-4 : because of linearity of E, $E(F) = 4E(N_1) + 3E(N_2) + 2E(N_3) + E(N_4) = 4+3+2+1 = \underline{10 = E(F)}$

2°) 2-5 : $Var(N_1 + N_2) = E((N_1+N_2)^2) - (E(N_1+N_2))^2 = E(N_1^2) + 2E(N_1N_2) + E(N_2^2) - [E(N_1)^2 + 2E(N_1)E(N_2) + E(N_2)^2]$

$\underline{Var(N_1 + N_2) = Var(N_1) + Var(N_2) + 2\,Covar(N_1, N_2)}$

and then, with $\begin{cases} Var(\alpha N) = \alpha^2 Var(N) \\ Covar(\alpha N, \beta N') = \alpha\beta\, Covar(N, N') \end{cases}$, $Var(F) = (16 + 9 + 4 + 1)\, Var(N_1) + (24+16+8+12+6+4)\, Covar(N_1 N_2)$

$\underline{Var(F) = 30\, Var(N_1) + 70\, Covar(N_1, N_2)}$

2-6 : Table of (N_1, N_2)

N_1 and N_2 following the same distribution, the table is symmetrical
$p((N_1, N_2) = (k, k')) = p((N_2, N_1) = (k, k'))$
$= \dfrac{C_4^{k} \, C_4^{k'} \, C_{44}^{13-k-k'}}{C_{52}^{13}}$

N_2 \ N_1	0	1	2	3	4
0	0,088	0,133	0,072	0,016	0,001
1	0,133	0,12	0,006	0,001	7.10^{-5}
2	0,072	0,006	0,011	$1{,}8.10^{-4}$	10^{-5}
3	0,018	0,001	$1{,}8.10^{-4}$	6.10^{-5}	$2{,}7.10^{-6}$
4	0,001	7.10^{-5}	10^{-5}	$2{,}7.10^{-6}$	$1{,}7.10^{-6}$

$E(N_1, N_2) = \sum_{k=0}^{4} \sum_{k'=0}^{4} k k' p(N_1 = k \cap N_2 = k') = 0{,}9412$

Therefore $Covar(N_1, N_2) = 0{,}9412 - 1^2 = -0{,}0588$ and $\underline{Var(F) = 17{,}06}$

N.B : It was beneficial to skip the computation which is not paid enough for the necessary time !

2-9: $F \sim \mathcal{N}(m=10;\ \sigma^2=17{,}06) \Rightarrow \dfrac{F-10}{4{,}13} \sim \mathcal{N}(0,1)$

$p(A) = p(F \leq 14{,}5) = p\left(\mathcal{N}(0,1) \leq \frac{14{,}5-10}{4{,}13} = 1{,}09\right) = \boxed{0{,}862 = p(A)}$

$p(B) = p(14{,}5 \leq F \leq 17{,}5) = p(1{,}09 \leq \mathcal{N}(0,1) \leq 1{,}815) = \boxed{0{,}103 = p(B)}$

$p(C) = p(F \leq 7{,}5) = 1 - p(\mathcal{N}(0,1) < 0{,}6) = \boxed{0{,}274 = p(C)}$

3°) for each (independent) deal $\begin{cases} 0{,}34 & \text{wins} \\ 0{,}66 & \text{looses} \end{cases}$ and if N = nb of deals he wins/5 $\sim \mathcal{B}(n=5, p=0{,}34)$
ind.

$E(N) = 1{,}7$ $\quad Var(N) = 1{,}12$

$p(A) = p(N \geq 3) = p(N=3) + p(N=4) + p(N=5) = \underline{0{,}22 = p(\text{passes})}$

B: he looses the first deal, therefore he has to win 3 deals out of 4 $(\mathcal{B}(n=4, p=0{,}34))$

$p(B) = C_4^3\, 0{,}34^3\, 0{,}66 + 0{,}34^4 = \underline{0{,}17 = p(B)}$

Q3 - Sharing out cards:

1°) After 8 tricks 5 cards/player $\rightarrow$ 10 cards for W and E ♠ Q 9 8 7 + 6 other cards (no matter which) ♢ and ♧

a) number of distributions E-W = number of hands for W $= C_{10}^5 = 252$.

b) E_W = "W has Q♠ and at least two other ♠"

$p(2 \text{ other } ♠) = \frac{1}{252}\left(C_3^2 \times C_6^2\right)$ $\quad p(3 \text{ other } ♠) = \frac{1}{252}\left(C_6^1\right) \Rightarrow p(E_W) = \frac{3\times 15 + 6}{252} = \frac{17}{84}$

$p(\text{one opponent has} \ldots) = p(E_W) + p(E_E) = 2 \times \frac{17}{84} = \frac{17}{42} = 0{,}4$

c) $p(\text{Cora wins}) = 1 - 0{,}4 = \underline{0{,}6 = p(\text{C. wins})}$

2°) a) W had 7 ♡ and 6 other cards at the beginning
has 5 cards/13 left $\Big\} \rightarrow R \sim \mathcal{H}(N=13, n=5, p=\frac{7}{13})$

$\forall k \in \{0, \ldots, 5\}\ \ p(R=k) = \dfrac{C_7^k\, C_6^{5-k}}{C_{13}^5}$

<u>Numerical values:</u>

$k=$	0	1	2	3	4	5
$p(R=k)$	0,047	0,081	0,32	0,408	0,163	0,016

b) Cora looses if one of her opponents has Q♠ and at least two other ♠

$k=5 \rightarrow$ E has all the ♠ $\rightarrow$ Cora looses

$k=4$ — $\frac{1}{6}$ W has the Q♠ $\rightarrow$ Cora wins $\rightarrow 0{,}163 \times \frac{1}{6}$
— $\frac{5}{6}$ not — looses

$k=3$ — $\frac{3\times 3}{C_7^3}$ EW has Q and two ♠
— $\frac{3}{C_7^3}$ E has Q — three ♠ $\Big\}$ Cora looses
— else Cora wins $\rightarrow 0{,}175$

$k=2$ — $\frac{3}{C_8^3}$ W has Q♠ and 2 ♠
— E has — — 3 ♠
— else $\rightarrow 0{,}211$

$k=1$ ... $\rightarrow 0{,}004$

$k=0$ same as question b) $\rightarrow 0{,}6 \times 0{,}047$

$\Bigg\}\ p(\text{Cora wins}) = 0{,}44.$