MIDTERM

An electronic calculator, the distributions tables as well as a sheet of personnal notes are allowed for the exam. Any kind of dictionary are permitted

USE ONE COPY BY EXERCISE

Exercise 1

A word of 5 letters is a list of 5 elements from a set of 26 letters (6 vowels and 20 consonants). As we are doing mathematics, we do not care if the word is meaningless (i.e. ZZZZZ is a 5 letters word). Compute the number of 5 letters words in the following situations :

1. *Without restriction.*

2. *With 5 distinct letters.*

3. *Starting with a consonant and finishing with a vowel.*

4. *Starting and finishing by a consonant.*

5. *With three times the letter e.*

6. *With the sequence tt (and no more than two t in the word).*

7. *With only one letter.*

8. *With two and only two different letters.*

Exercise 2

Let us consider a population E of N people. A repeated trial is performed according to the following process :

- *Step 1 : Each person of E takes a test. The probability of success is p. The result of each individual are mutually independent. The people who fail are eliminated.*
- *Step 2 : We start over with the people not eliminated after step 1. An other test is performed with same probabilty p of success and the people who fail are elimindated.*
- *We keep going (step $3, 4, \dots,$) until everyone is eliminated.*

Let X_n be the random variable which counts the number of remaining people after step n. For $n = 0$ we have $X_0 = N$.

1. *What is the probability distribution of X_1 ?*

2. *Study of X_n :*

  a. *At the end of step $n-1$ we assume there are h individual left (i.e. $X_{n-1} = h$). Compute $P(X_n = k | X_{n-1} = h)$ (you may need to consider the cases $h < k$ and $h \geqslant k$).*

  b. *Using the law of total probability give an expression for $P(X_n = k)$ in terms of $P(X_{n-1} = h)$ for $0 \leqslant h \leqslant N$.*

  c. *Prove that $\binom{N}{h}\binom{h}{k} = \binom{N}{k}\binom{N-k}{h-k}$.*

  d. *Prove by induction that $X_n \sim \mathcal{B}(N, p^n)$. What are $E(X_n)$ and $V(X_n)$?*

3. *We now assume $p = \frac{5}{6}$ (for instance E is a population of N players. Each player tosses a die and is eliminated if he gets 6). After how many steps do we have in average at most one individual left ? Do the calculation for $N = 10, 20, 100, 1000$.*

Please Turn Over !

Exercise 3

Let X_1 and X_2 two independent random variables with $X_i \sim \mathcal{E}(\lambda)$.

1. *Prove that $P(X_i > t) = e^{-\lambda t}$.*

2. *Let $Y = min(X_1, X_2)$.*

- **a.** *Compute $P(Y > t)$.*
- **b.** *Deduce the density function of Y and conclude that $Y \sim \mathcal{E}(2\lambda)$.*
- **c.** *Give $E(Y)$ and $V(Y)$*

3. *Let $Z = max(X_1, X_2)$.*

- **a.** *Give the expression $P(Z > t)$ in terms of $P(X_1 > t)$, $P(X_2 > t)$ and $P(Y > t)$.*
- **b.** *Prove the density function of Z is $f_Z(t) = 2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t}$ when $t \geqslant 0$, and $f_Z(t) = 0$ if $t < 0$.*
- **c.** *Calculate $E(Z)$.*

4. **[extra-points]** *Let $W = X_1 + U$ where $U \sim \mathcal{E}(2\lambda)$. We assume X_1 and U are independent.*

- **a.** *Compute the density function of (X_1, U).*
- **b.** *Determine the density function of W.*
- **c.** *Compare the distributions Z and W.*
- **d.** *Calculate $V(Z)$.*

5. *Application : Let us consider an electronic circuit with two components $C1$ and $C2$. We denote by X_1 and X_2, the lifetime (in one hour units) of components 1 and 2. We assume X_1 and X_2 follow the exponential distribution $\mathcal{E}(0.001)$. We consider two ways of connecting the components :*

- *In series (see figure 1) : the circuit is working as long as both $C1$ and $C2$ are not out of order.*
- *In parallel (see figure 2) : the circuit is working as long as one of the component is functioning.*

FIG. 1 – Series Circuit

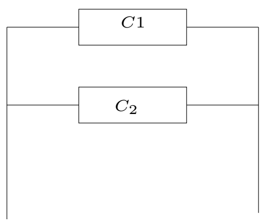

FIG. 2 – Parallel circuit

- **a.** *In the series circuit find the probability that the circuit lasts between 1500 and 2000 hours.*
- **b.** *Same question with the parallel circuit.*